THE UNITED REPUBLIC OF TANZANIA
MINISTRY OF EDUCATION, SCIENCE AND TECHNOLOGY
NATIONAL EXAMINATIONS COUNCIL OF TANZANIA

CANDIDATES' ITEM RESPONSE ANALYSIS REPORT ON THE ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION (ACSEE), 2021

ADVANCED MATHEMATICS

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CANDIDATES' ITEM RESPONSE ANALYSIS REPORT FOR THE ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION (ACSEE) 2021

142 ADVANCED MATHEMATICS

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FOREWORD

This report is based on the performance of candidates who sat for the Advanced Certificate of Secondary Education Examination (ACSEE) 2021 in Advanced Mathematics. The report aims at providing feedback to students, teachers, parents, educational administrators, school managers, policy makers and the general public about candidates' abilities in the Advanced Mathematics subject.

The performance in the ACSEE 2021 Advanced Mathematics was good as 11,879 candidates (93.49%) out of 12,706 passed the examination. This good performance was observed in 15 out of the 18 topics examined. Candidates had good performance in the topic of Sets, Numerical Methods, Linear Programming, Coordinate Geometry II, Functions, Hyperbolic Functions, Complex Numbers, Trigonometry, Statistics, Logic, Calculating Devices, Algebra, Differential Equations, Coordinate Geometry I and Probability. The candidates' performance was average in the topic of Vectors and weak in Integration and Differentiation.

The candidates' performance was weak due to failure of candidates to do the following: use the integration by parts method in integrating the product of algebraic and trigonometric functions; use correctly the substitution rule to compute definite integrals; use turning points in sketching the graph of polynomial functions and find the Taylor's series of radical functions.

The council expects that the feedback provided in this report will enable the education stakeholders to identify proper measures to improve future performance in this subject.

The Council would like to thank all those who participated in processing and analyzing the data used in this report.

Dr. Charles E. Msonde
EXECUTIVE SECRETARY

1.0 INTRODUCTION

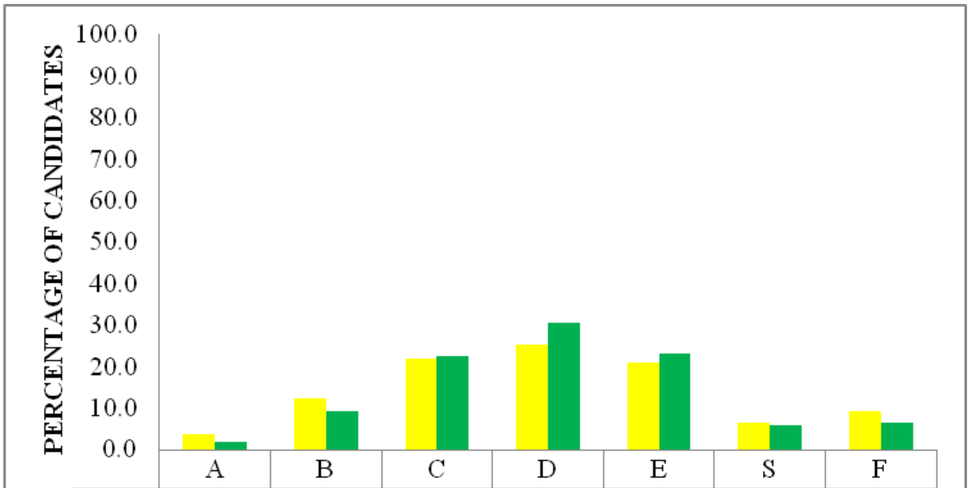
This report provides feedback to education stakeholders on how the candidates performed in Advanced Mathematics subject in the Advanced Certificate of Secondary School Examination 2021.

Particularly, the report analyses the candidates' performance in all topics examined in Advanced Mathematics. The Advanced Mathematics examination had two papers: paper 1 and paper 2. In paper one, there were ten (10) compulsory questions where each question carried ten (10) marks. Paper two consisted of four (4) compulsory questions in section A where each question carried fifteen (15) marks and four (4) optional questions in section B from which the candidates were required to answer any two. In section B each question carried twenty (20) marks.

The analysis of the individual questions comprises a brief account of the requirements of the questions and the performance of candidates. Figures and charts have been used to summarise the performance. The factors that accounted for good, average and weak performance in each question have been indicated and illustrated using samples of candidates' responses which are inserted as extracts.

The analysis of the candidates' performance in each topic has been done to identify topics with good, average and weak performance. Three colours have been used to signify performance whereby green colour stands for good performance, yellow colour for average performance and red colour for weak performance. The percentage boundaries 0-34, 35-59 and 60-100 are used to represent weak, average and good performance respectively.

A total of 12,706 candidates sat for the Advanced Mathematics Examination, out of whom 11,879 (93.49%) candidates passed. This performance is better than that of 2020 where 90.63 per cent of 10,125 candidates passed. This represents an increase of 2.86 per cent in the candidates' performance. The candidates who passed the Advanced Mathematics examinations obtained grades ranging from A to F as indicated in Figure 1.



	A	B	C	D	E	S	F
2020	3.6	12.4	21.9	25.3	21.0	6.5	9.4
2021	1.9	9.4	22.5	30.5	23.3	6.0	6.5

Figure 1: *Overall Performance in the 2020 and 2021 Advanced Mathematics Examinations*

Finally, the conclusion and some recommendations have been given at the end of the report. The recommendations made will help teachers and the government to improve candidates' performance in future in Advanced Mathematics examinations.

2.0 ANALYSIS OF CANDIDATES' PERFORMANCE IN EACH QUESTION

2.1 142/1 ADVANCED MATHEMATICS 1

2.1.1 Question 1: Calculating Device

This question had parts (a) and (b). In part (a), the candidates were instructed to use a non-programmable calculator to compute: (i) the value of $\sqrt[x]{\ln(\log x)^4 + e^{5x}\sin 3x}$ correct to 7 decimal places for $x = 0.2$ radian; (ii) the derivative of $\left(x^2 - x + 4\right)^6$ at $x = 5$ and the modulus of $\frac{(4+3i)\times(3+4i)}{(3+i)}$ in radian giving the answer to four significant figures.

In part (b), the candidates were given that the weights x of 10 insects in mg are 1.20, 0.04, 1.40, 0.04, 0.716, 0.17, 1.20, 1.20, 2.40 and 3.00 respectively. They were required to use a non-programmable calculator to compute $\sum x^2$ and δx correct to three decimal places.

The analysis of data shows that 12,601 candidates (98.7%) attempted this question. Among them, 8908 candidates (70.7%) scored marks ranging from 3.5 to 10. Therefore, the candidates' performance in this question was good. Figure 2 shows the percentage of candidates who obtained weak, average and good performance.

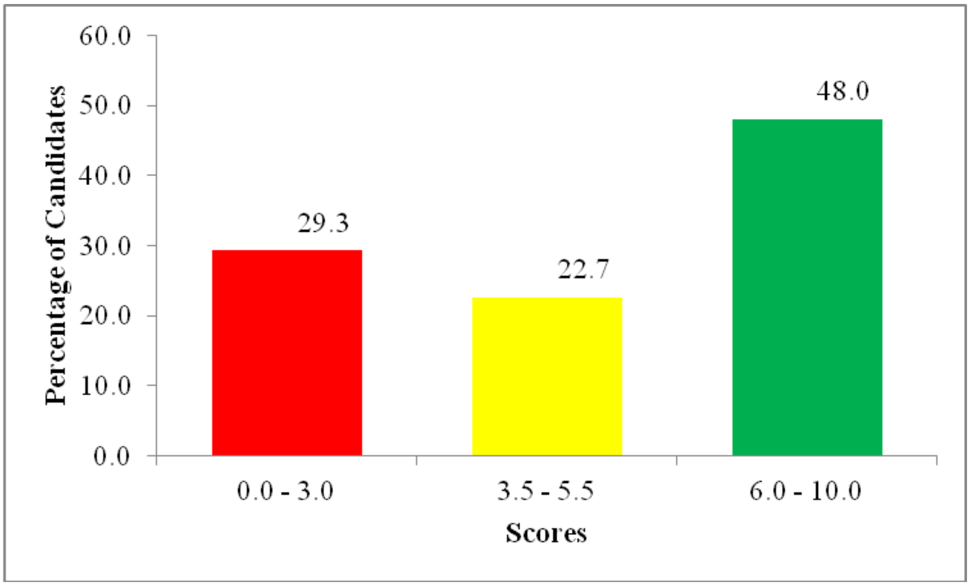
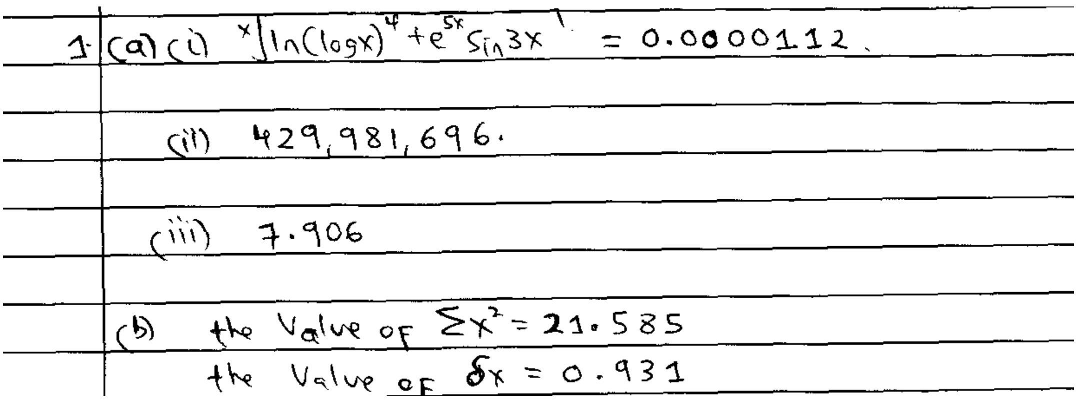


Figure 2: *Candidates' performance in question 1*

Based on Figure 2, 48.0 percent of the candidates scored high marks. These candidates were able to use the functional keys of a non-programmable calculator to compute $\sqrt[x]{\ln(\log x)^4 + e^{5x}\sin 3x}$ where $x = 0.2$ radian and got 0.0000112 correct to 7 decimal places as required in part (a) (i). Those who got all marks in part (a) (ii) plugged correctly the expression $\frac{d}{dx}\left(\left(x^2 - x + 4\right)^6\right)$ into the calculator and computed its derivative at $x = 5$ to get 429,981,696. In part (a) (iii), the candidates were able to plug the expression $\frac{(4+3i)\times(3+4i)}{(3+i)}$ into a non programmable calculator to get 2.5+7.5i. Then, they typed the real and imaginary parts of the resulting complex number into the calculator as $\sqrt{(2.5)^2 + (7.5)^2}$ giving the modulus equal to 7.906 correct to four significant figures as required.

The candidates who did part (b) plugged in correctly the given weights of 10 insects into the calculator and came out with the required statistical results as $\sum x^2 = 21.585$ and $\delta x = 0.931$. Extract 1.1 shows a sample of correct responses from one of the candidates.


1\. (a) (i) $\sqrt[x]{\ln(\log x)^4 + e^{5x}\sin 3x} = 0.0000112$.

(ii) 429,981,696.

(iii) 7.906

(b) the Value of $\sum x^2 = 21.585$

the Value of $\delta x = 0.931$

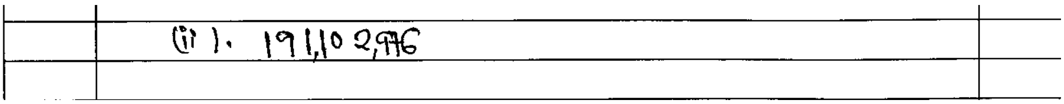
Extract 1.1: *A sample of correct response to question 1*

Extract 1.1 shows a response from a candidate who was able to accurately use the function keys of a calculator to compute the required values in each part.

Despite having good performance in this question, Figure 2 shows that, there were candidates (29.3%) who scored low marks. The analysis of procedures shows that in doing part (a) (i), these candidates did not set their calculators in radian mode to evaluate the given expression. Further analysis of their responses shows that they ended up getting incorrect answers such as -5.4579323 instead of 0.0000112.

Another common mistake was failure of candidates to express the given radical as $\left[\ln(\log 0.2)^4 + e^{5\times0.2}\sin(3\times0.2)\right]^{\frac{1}{0.2}}$. A number of candidates computed $\left[\ln(\log 0.2)^4 + e^{5\times0.2}\sin(3\times0.2)\right]^{0.2}$ to get 0.6337932 instead of 0.0000112. It was noted that several candidates were able to evaluate the given radical to get 0.000011186 but could not present their final answer in 7 decimal places as 0.0000112. The candidates who got wrong part (a) (ii) plugged into the calculator the expression $\left(x^2 - x + 4\right)^6$ instead of $\frac{d}{dx}\left[\left(x^2 - x + 4\right)^6\right]$ which gave 191,102,976 instead of 429,981,696. In part (a) (iii), most candidates computed the modulus for the given expression as 7.90569415 but presented it in 4 decimal places as 7.9057 instead of 4 significant figures as 7.906. Several of these candidates rationalized the given expression to get the complex number $2.5 + 7.5i$ but did not proceed further with the computation of the required modulus in four significant figures.

The candidates, who got incorrect answers in part (b), did not express the final answers correct to three decimal places. As a result, they wrote $\sum x^2 = 21.584756$ and $\delta x = 0.930922145$ instead of 21.585 and 0931 respectively. Extract 1.2 illustrates a sample solution of a candidate incorrect answer. The answer shows the candidate that lacked knowledge and skills on the concepts tested in part (a) (ii) of this question.



Extract 1.2: *A sample of an incorrect response to Question 1*

In extract 1.2, the candidate evaluated $\left(x^2 - x + 4\right)^6$ at $x = 5$ contrary to the requirements which required the candidate to plug in the given expression by using the derivative key.

2.1.2 Question 2: Hyperbolic Functions

This question consisted of parts (a), (b), and (c). In part (a), the candidates were required to express $4\cosh\theta + 5\sinh\theta$ in the form $R\sinh(\theta+\alpha)$ and then to find the values of $R\sinh(\theta+\alpha)$, R and $\tanh\alpha$. In part (b), the candidates were instructed to show that $\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\}$ and in part (c), they were required to find $\frac{dy}{dx}$ given that $y = \frac{\cosh 2x}{1 + \sinh 2x}$.

The analysis shows that 13.2 percent of the candidates who attempted this question scored from 0 to 3 marks, 25.4 percent from 3.5 to 5.5 and 61.5 percent from 6 to 10 marks. Generally, the candidates' performance in this question was good as 86.9 percent of the candidates got more than 3 marks. Figure 3 illustrates the candidates' performance in this question.

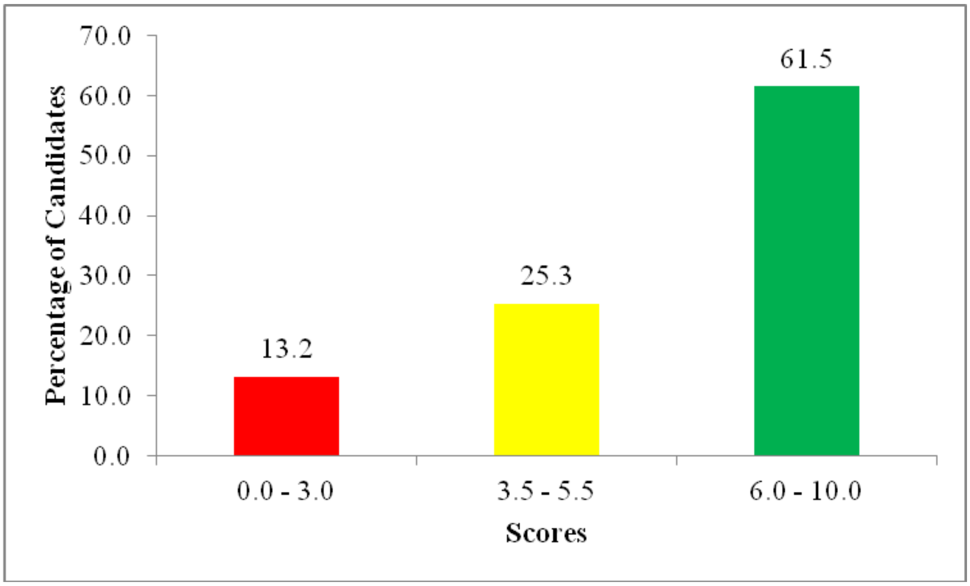


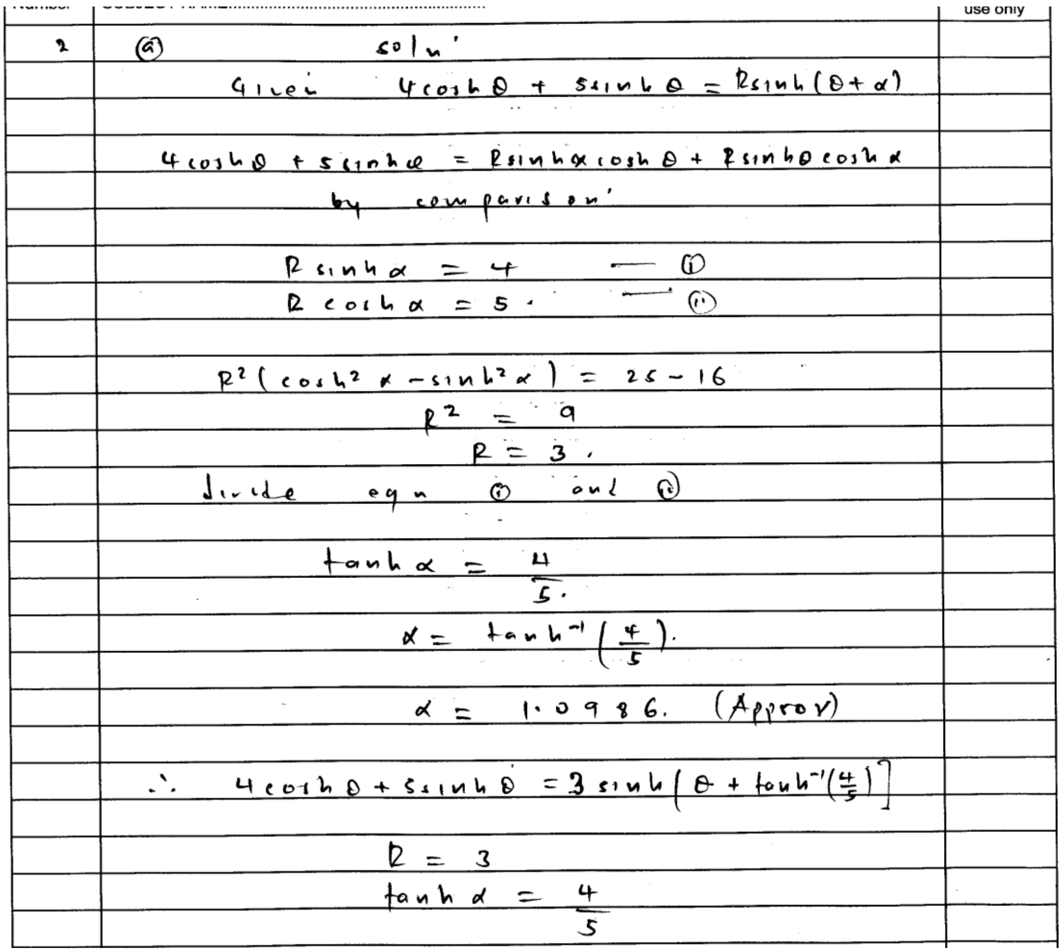
Figure 3: *Candidates' performance in question 2*

The candidates (61.5%) who scored high marks were able to expand $R\sinh(\theta + \alpha)$ in part (a) as $R\sinh\theta\cosh\alpha + R\cosh\theta\sinh\alpha$ and compare it with $4\cosh\theta + 5\sinh\theta$ to get $\alpha = 1.0986$, $R = \pm 3$ and hence wrote $3\sinh(\theta + 1.0986)$ which was the required form.

In part (b), they demonstrated the following strength: they equated $\cosh^{-1} x$ with y implying that $\cosh y = x$; they substituted $\cosh y$ with $\cosh y = \frac{e^y + e^{-y}}{2}$ in the equation $\cosh y = x$ to get $x = \frac{e^y + e^{-y}}{2}$ which was solved to prove that $\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\}$. Other candidates substituted $\cosh y = x$ in the identity $\cosh^2 y - \sinh^2 y = 1$ to get $\sinh y = \sqrt{x^2 - 1}$. Thereafter, they replaced

$\cosh y$ and $\sinh y$ in the identity $\sinh y + \cosh y = e^y$ with x and $\sqrt{x^2 - 1}$ giving $\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\}$ as required.

In part (c), the candidates were able to do the following: multiply $1 + \sinh 2x$ with the derivative of $\cosh 2x$ to get $2\sinh 2x + 2\sinh^2 2x$; multiply $\cosh 2x$ with the derivative of $1 + \sinh 2x$ to get $2\cosh^2 2x$; subtract $2\cosh^2 2x$ from $2\sinh 2x + 2\sinh^2 2x$ to get $2(\sinh 2x - 1)$ and divide $2(\sinh 2x - 1)$ by the square of $1 + \sinh 2x$ to get $\frac{dy}{dx} = \frac{2(\sinh 2x - 1)}{(1 + \sinh 2x)^2}$. This shows that they were knowledgeable about the quotient rule which says that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator. Extract 2.1 shows a sample from one of the candidates who attempted this question correctly.


2 (a) soln'

Given $4\cosh\theta + \sinh\theta = R\sinh(\theta + \alpha)$

$4\cosh\theta + \sinh\theta = R\sinh\alpha\cosh\theta + R\sinh\theta\cosh\alpha$

by comparison

$R\sinh\alpha = 4$ —— (i)

$R\cosh\alpha = 5$ —— (ii)

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 25 - 16$

$R^2 = 9$

$R = 3$

divide eqn (i) and (ii)

$\tanh\alpha = \frac{4}{5}$

$\alpha = \tanh^{-1}\left(\frac{4}{5}\right)$

$\alpha = 1.0986$ (Approx)

$\therefore 4\cosh\theta + \sinh\theta = 3\sinh\left[\theta + \tanh^{-1}\left(\frac{4}{5}\right)\right]$

$R = 3$

$\tanh\alpha = \frac{4}{5}$

2	(b)	soln'	
		To show	
		$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$.	
		let $y = \cosh^{-1} x$	
		$\cosh y = x$	
		by definition'	
		$\cosh y = \frac{e^y + e^{-y}}{2}$	
		$e^{y} + e^{-y} = 2x$	
		$e^{2y} - 2xe^y + 1 = 0$	
		on solving	
		$e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	
		$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$	
		$e^y = \frac{2x \pm 2\sqrt{x^2-1}}{2}$	
		$e^y = x \pm \sqrt{x^2-1}$	
		neglect -ve value	
		$e^y = x \pm \sqrt{x^2-1}$	
		$\ln e^y = \ln(x + \sqrt{x^2-1})$	
		$y = \ln(x + \sqrt{x^2-1})$	
		$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ Hence shown'	

2	(c)	soln	
		Given	
		$y = \frac{\cosh 2x}{1 + \sinh 2x}$	
		by quotient rule	
		$\frac{d(v/u)}{dx} = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{u^2}$.	

$$\frac{dy}{dx} = \frac{(1+\sinh 2x)(2\sinh 2x) - \cosh 2x(2\cosh 2x)}{(1+\sinh 2x)^2}$$

$$\frac{dy}{dx} = \frac{2\sinh 2x - 2(-\sinh^2 2x + \cosh^2 2x)}{(1+\sinh 2x)^2}.$$

$$\frac{dy}{dx} = \frac{2\sinh 2x - 2(\cosh^2 2x - \sinh^2 2x)}{(1+\sinh 2x)^2}$$

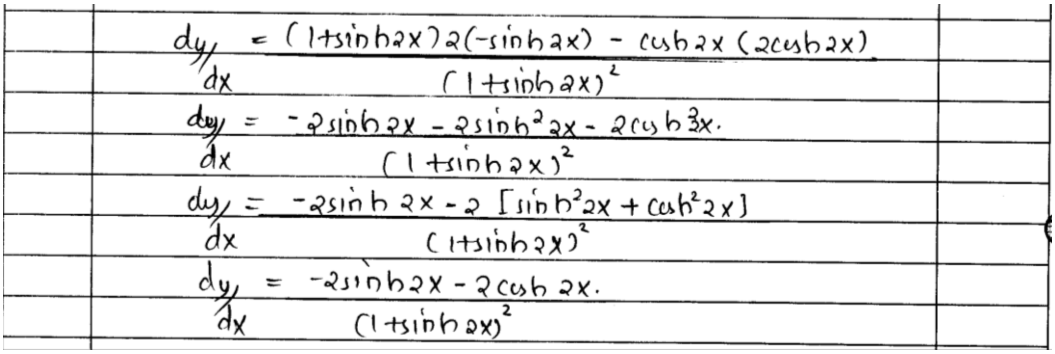
$$\frac{dy}{dx} = \frac{2(\sinh 2x - 1)}{(1+\sinh 2x)^2}.$$

$$\therefore \frac{dy}{dx} = \frac{2(\sinh 2x - 1)}{(1+\sinh 2x)^2}$$

Extract 2.1: *A correct response to Question 2*

In Extract 2.1, the candidate was able to express the given function in the required form in part (a). In part (b), the candidate had adequate knowledge and skills on how to derive the inverse hyperbolic cosine. In part (c), the candidate also correctly used the quotient to differentiate the given hyperbolic functions.

On the other hand, a few candidates (13.2%) who scored low marks had some challenges as follows: In part (a), some of them wrote incorrect forms for the expression $R\sinh(\theta+\alpha)$ such as $R\sinh\theta\cosh\alpha - \text{R}\cosh\theta\sinh\alpha$ and $R\cosh\theta\cosh\alpha + \text{R}\sinh\theta\sinh\alpha$ instead of $R\sinh\theta\cosh\alpha + \text{R}\cosh\theta\sinh\alpha$. Others simplified the expression using wrong identities $\cosh^2\alpha + \sinh^2\alpha = 1$ to get $R = \sqrt{41}$. Several candidates worked out an expression $\frac{\cosh\alpha}{\sinh\alpha}$ to get $\tanh\alpha = -0.5108$ instead of $\tanh\alpha = 1.0986$. In part (b), they could not derive the inverse hyperbolic cosine correctly. It was noted that they used wrong identities like $\cosh^2 y + \sinh^2 y = 1$ and $\cosh y = \frac{e^y - e^{-y}}{2}$ instead of $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y = \frac{e^y + e^{-y}}{2}$ which were necessary steps in proving the given function. As a result, they ended up with incorrect inverse hyperbolic cosine that is $\cosh^{-1} x = \ln\left\{x + \sqrt{x^2+1}\right\}$ instead of $\cosh^{-1} x = \ln\left\{x + \sqrt{x^2-1}\right\}$. In part (c), they applied incorrect rules like $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ $\frac{dy}{dx} = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{v^2}$, and $\frac{dy}{dx} = \frac{u\frac{dv}{dx} + v\frac{du}{dx}}{v^2}$. Others used the concept of differentiating implicitly function contrary to the requirement of the question. These candidates lacked knowledge and skills on manipulation of the quotient rule to differentiate hyperbolic functions. Extract 2.2 is a sample solution from a candidate who did part (c) of this question incorrectly.



$$\frac{dy}{dx} = \frac{(1+\sinh 2x)2(-\sinh 2x) - \cosh 2x\,(2\cosh 2x)}{(1+\sinh 2x)^2}$$

$$\frac{dy}{dx} = \frac{-2\sinh 2x - 2\sinh^2 2x - 2\cosh 3x.}{(1+\sinh 2x)^2}$$

$$\frac{dy}{dx} = \frac{-2\sinh 2x - 2[\sinh^2 2x + \cosh^2 2x]}{(1+\sinh 2x)^2}$$

$$\frac{dy}{dx} = \frac{-2\sinh 2x - 2\cosh 2x.}{(1+\sinh 2x)^2}$$

Extract 2.2: *An incorrect response to Question 2 (c)*

In Extract 2.2, the candidate confused the derivative of trigonometric cosine with the derivative of hyperbolic cosine as he/she obtained $\frac{d}{dx}(\cosh 2x)$ equal to $-\ 2\sinh 2x$ instead of $2\sinh 2x$.

2.1.3 Question 3: Linear Programming

The question comprised parts (a) and (b). Part (a) of this question read as "A farmer needs 10 kg of SA and 15 kg of CAN. He can buy bags containing 2 kg of SA and 1 kg of CAN or he can buy tins containing 1 kg of SA and 3 kg of CAN. If the cost of each bag and tin are 20/= and 50/= respectively." The candidates were required to write down four inequalities representing the problem. In part (b), they were instructed that: Mr Chapakazi has two storage depots. He stores 200 tons of rice at depot 1 and 300 tons at depot 2. The rice has to be sent to three marketing centres A, B and C. The demands at A, B and C are 150, 150 and 200 tons respectively. The transport cost per ton to each marketing centre is as shown in the following table 1:

Table 1: The cost of transport cost per ton of rice

Depots	**Marketing Centres**		
	A	B	C
Depot 1	50	100	70
Depot 2	80	150	40

The candidates were required to find the quantity of rice to be sent to each marketing centre so that the transportation cost is minimum.

The analysis of data shows that 10.3 percent of the candidates who attempted this question scored marks ranging from 0 to 3, 26.4 percent scored marks ranging from 3.5 to 5.5 and 63.3 percent scored 6 marks and above. Generally, the candidates' performance in this question was good as 89.7 percent of the candidates got at least 3.5 marks. Figure 2 illustrates the candidates' performance in this question.

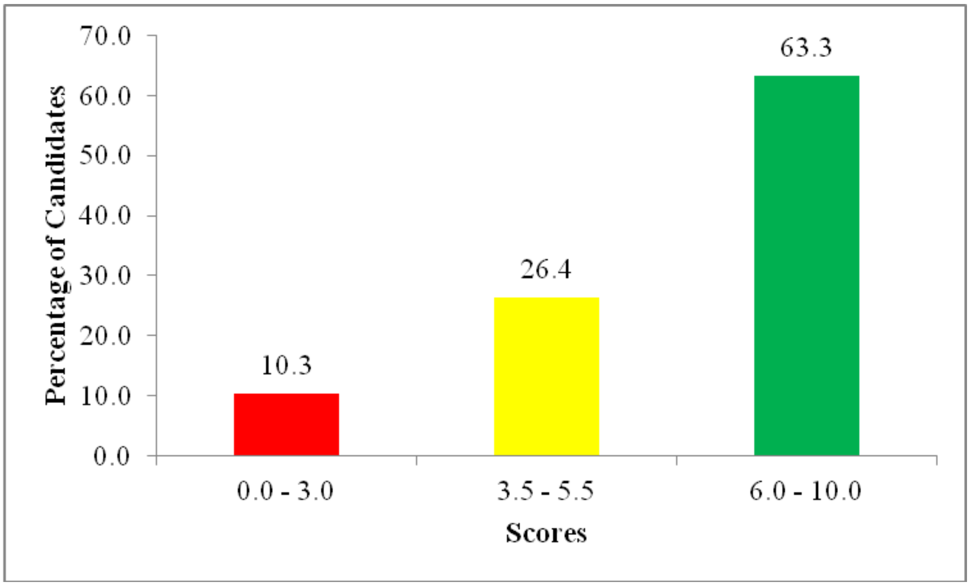


Figure 4: *Candidates' performance in question 3*

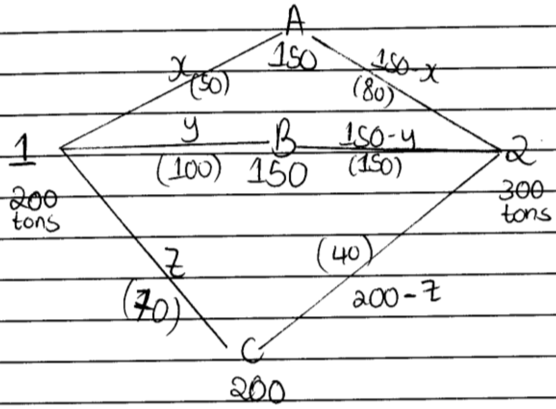
The candidates who scored full marks were able to write down the correct inequalities representing the given linear programing problem using x as the number of bags and y as a number of tins of rice. In part (a), the inequalities were

$2x+y\geq 10$, $x+3y\geq 15$ $x\geq 0$ and $y\geq 0$. In part (b), the candidates with the aid of diagrams managed to formulate the constraints from transportation problem. Those inequalities were $x\leq 150$, $y\leq 150$, $x+y\leq 200$, $x\geq 0$ and $y\geq 0$. They were also able to formulate the objective function, which was simplified to $f(x,y)=48{,}500-60x-80y$. Then these candidates represented the inequalities graphically, indicated the feasible region correctly and got the corner points $A(0,0)$, $B(150,0)$, $C(150,50)$, $D(50,150)$ and $E(0,150)$. Thereafter, they substituted the values of the corner points in the objective function to get 48,500/=, 39,500/=, 35,500/=, 33,500/= and 36,500/= respectively. Finally, they concluded that the minimum cost occurs at the point (50,150) giving 50, 100, 150, and 200 tons to be transported from depot 1 to A, depot 2 to A, depot 1 to B and depot 2 to C respectively. This indicates that the candidates had adequate knowledge and skills on how to transform word and transportation problems into mathematical models as well as solving transportation problem graphically. Extract 3.1 is a sample response from a candidate who answered this question correctly.

03. (a) let
'x' be the number of bags bought
'y' be the number of tins bought

Constraints
$2x + y \geq 10$
$x + 3y \geq 15$
$x \geq 0$
$y \geq 0$
objective function
$f(x,y) = 20x + 50y$

03(b) let
"x" be the tons of rices from Depot 1 to A
"y" be the tons of rices from Depot 1 to B
"z" be the tons of rices from Depot 1 to C



Constraints

$x \geq 0$ --- (i)

$y \geq 0$ --- (ii)

$z \geq 0$

but

$x + y + z = 200$

$z = 200 - (x+y)$

$200 - (x+y) \geq 0$

$x + y \leq 200$ --- (iii)

$x \leq 150$ --- (iv)

$y \leq 150$ --- (v)

$x + y \geq 0$ --- (vi)

Objective function

$f(x,y) = 50x + 100y + 70z + 80(150-x) + 150(150-y) + 40(200-z)$

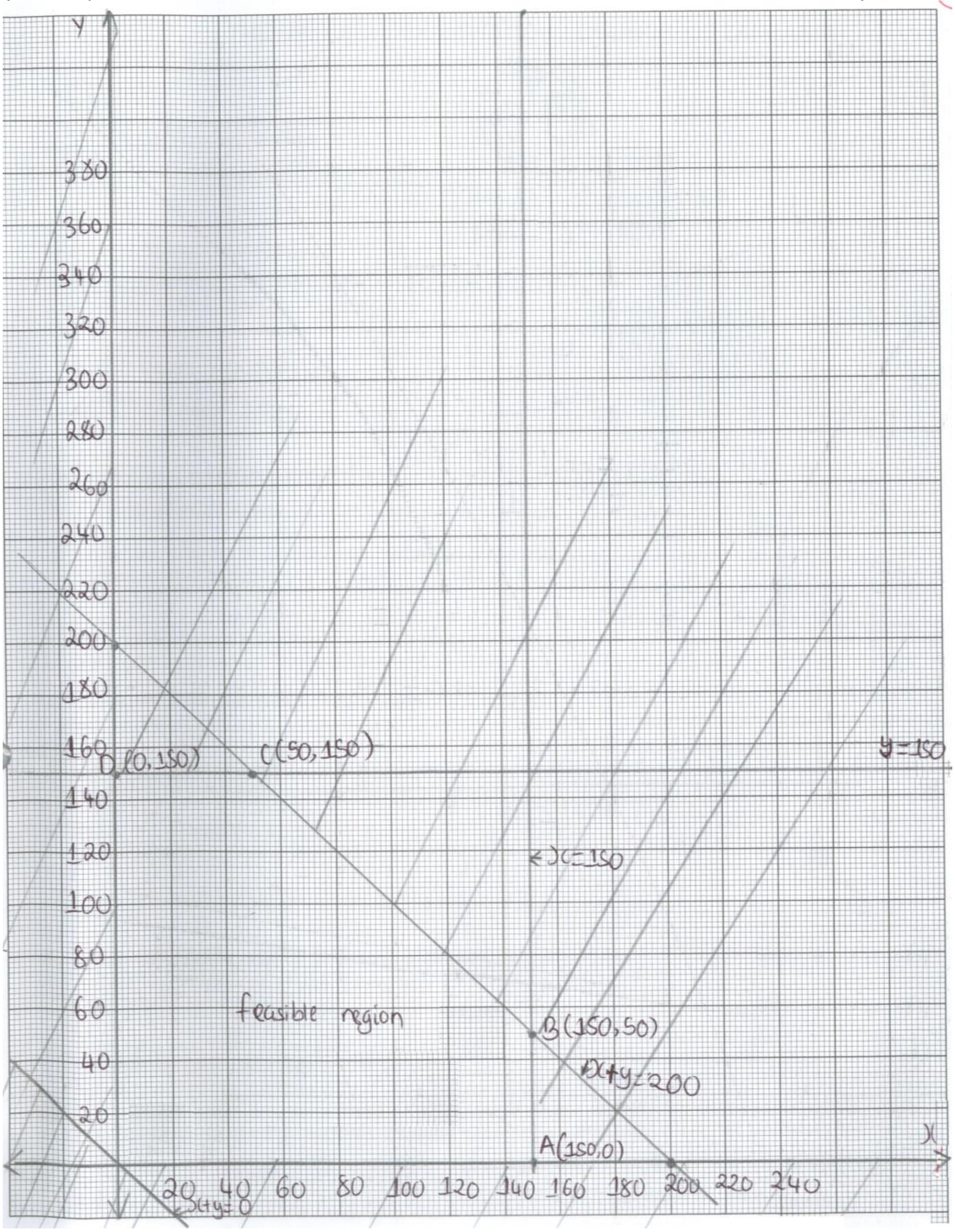
$= -30x - 50y + 30z + 42500$

$= -30x - 50y + 30(200-x-y) + 42500$

$f = -60x - 80y + 48500$

Corner points	$f = -60x - 80y + 48500$
A(150,0)	39500
B(150,50)	35500
C(50,150)	33500
D(0,150)	36500

∴50 tons should be sent from Depot 1 to A, 150 tons from Depot 1 to B, 100 tons from Depot 2 to A and 200 from Depot 2 to C to make a minimum cost of 33500/=



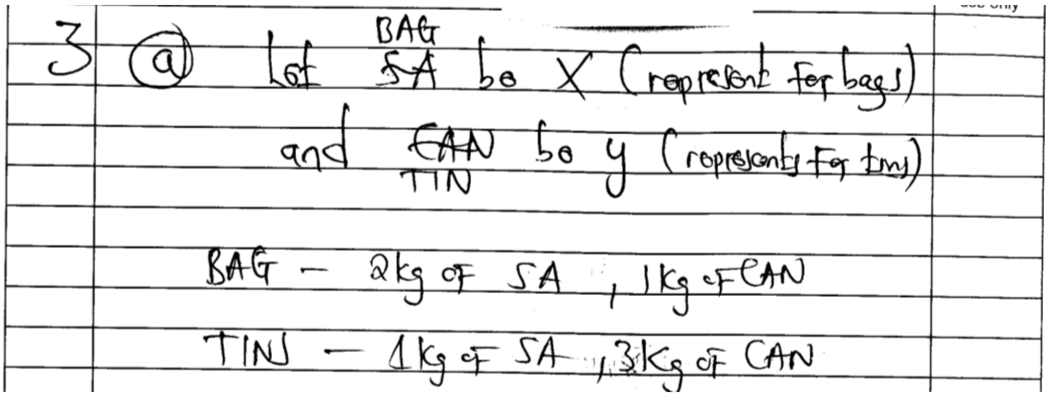
Extract 3.1: *A correct response to Question 3*

In Extract 3.1, the candidate was able to transform the word problem into the required inequalities in part (a). The candidate was also able to find the number of tons of rice to be transported from depot 1 and 2 to each marketing centre; A, B, and C in part (b).

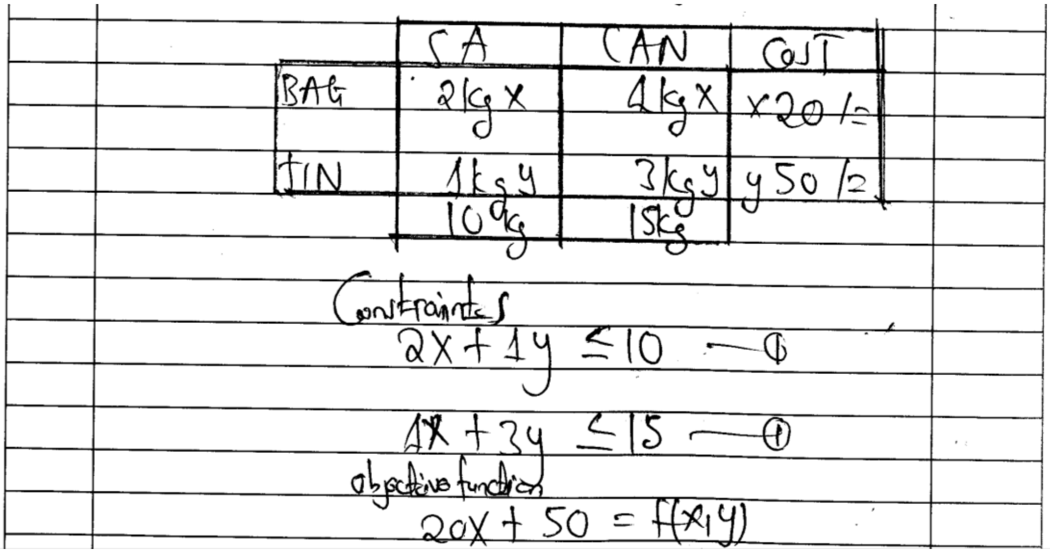
On the other hand, there were few candidates (10.3%) who got the question wrong and the analysis of their work shows that they faced some challenges as follows: In part (a), they incorrectly interchanged the inequality signs. Due to this reason they wrote $2x+y\leq 10$, $x+3y\leq 15$ $x\leq 0$ and $y\leq 0$ instead of $2x+y\geq 10$, $x+3y\geq 15$ $x\geq 0$ and $y\geq 0$. Some of them also confused between inequalities and equations. As a result they wrote $2x+y=10$ and $x+3y=15$. Other incorrect inequalities observed were $x\leq 10$, $y\leq 15$, $x\geq 2$, $y\geq 1$, $x\leq 1$, $2x\leq 10$, $y\leq 3$, $3y\leq 15$ and $1\leq y\leq 3$. This shows that the candidates did not understand how to transform word problems into mathematical models.

In part (b), a number of candidates failed to present the given transportation problem diagrammatically and were therefore unable to formulate the constraints and the objective function. Some candidates also treated the transportation problem as a normal linear programming problem and wrote the constraints $50x+80y\geq 150$, $100x+150y\geq 150$, $70x+40y\geq 200$ and $f(x,y)=200x+300y$. Other common errors seen were reversing the inequality sign $\leq$ with $\geq$, not indicating the correct feasible region and failure to label the graphs.

In general, the candidates who got incorrect answers lacked knowledge and skills on how to transform transportation problems into mathematical models and solve them graphically. Extract 3.2 is a sample of responses from a candidate who gave incorrect answers to part (a) of this question.


3 (a) Let BAG be X (represent for bags)
and TIN be y (represents for tins)

BAG – 2kg of SA , 1kg of CAN
TINS – 1kg of SA , 3kg of CAN



	SA	CAN	COST
BAG	2kg x	4kg x	x 20/=
TIN	1kg y	3kg y	y 50/=
	10kg	15kg	

Constraints

$2x + 1y \leq 10$ — (i)

$4x + 3y \leq 15$ — (ii)

objective function

$20x + 50 = f(x,y)$

Extract 3.2 *A sample of an incorrect responses to Question 3 (a)*

In Extract 3.2, the candidate was not able to formulate the required inequalities. He/she reversed the inequality signs and did not include the non negative constraints like $x, y \geq 0$.

2.1.4 Question 4: Statistics

The question comprised of parts (a) and (b). In part (a), the candidates were given the numbers $x_1, x_2, x_3, ..., x_n$ with a standard deviation of 10, and were required to find the standard deviation of numbers $(2x_1 + 1), (2x_2 + 1), ..., (2x_n + 1)$. In part (b), the candidates were given the information that "A classroom teacher measures lengths of 50 students to the nearest centimetre" whereby the results are summarized in the table 2.

Table 2: Lengths of 50 students to the nearest centimetre

Length (cm)	31 – 35	36 – 40	41 – 45	46 – 50	51 - 55	56 – 60
Frequency (f)	3	6	17	10	9	5

The candidates were required to calculate (i) the first and third quartiles correct to two decimal places and (ii) the 70th percentile correct to one decimal place.

The analysis of data shows that the question was attempted by 12,396 (97.1%) candidates, out of which 80.3 percent of the candidates scored more than 3 marks.

Further analysis shows that, 19.7 percent of the candidates scored 0 to 3 marks, 61.5 percent scored from 3.5 to 5.5 marks and 18.8 percent scored from 6.0 to 10.0 marks. The data also shows that 476 (3.8%) candidates scored 10.0 marks while 308 (2.5%) candidates scored 0. Therefore, the candidates' performance in this question was good. The summary of candidates' performance is presented in Figure 5.

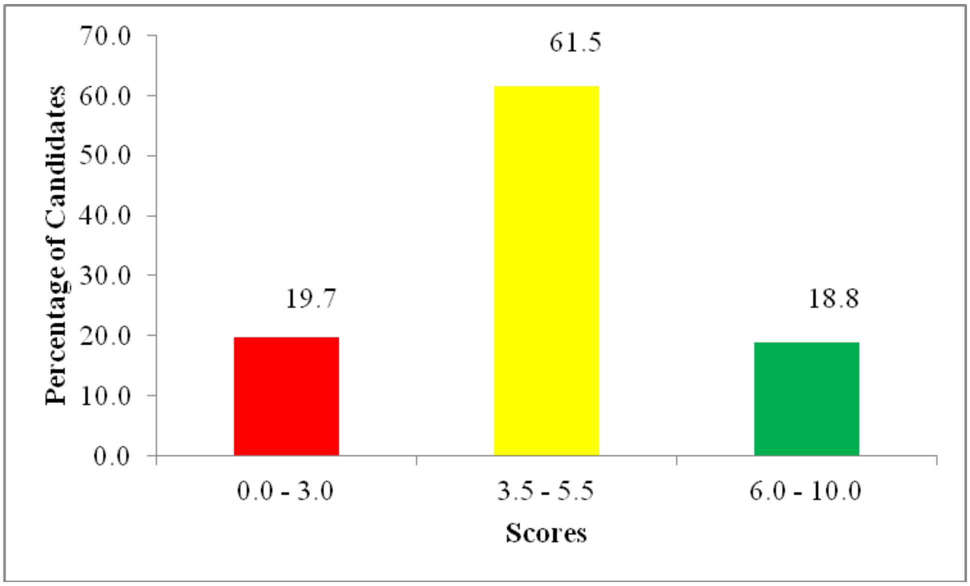


Figure 5: *Candidates' performance in question 4*

The candidates who responded to part (a) correctly were able to show that the new mean $\bar{x}'$ for the sequence $(2x_1+1), (2x_2+1), \ldots\ldots, (2x_n+1)$ is given by $2\bar{x}+1$ where $\bar{x} = \frac{x_1 + x_2 + \ldots\ldots, x_n}{N}$; and to simplify the expression $\sqrt{\frac{\sum_{i=1}^{n}(2x_i + 1 - \bar{x}_1)^2}{N}}$ to $2\sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{N}}$ so that the new standard deviation (d') of the sequence of numbers is equal to $2 \times 10 = 20$. Other candidates opted to use the fact that $\operatorname{var}(ax + b) = a^2 \operatorname{var}(x)$ where $a = 2$ and $\operatorname{var}(x) = 10^2$ and got the same value of the new standard deviation as those who decided to apply the first technique.

In part (b), the candidates managed to: construct the frequency distribution table with the column of cumulative frequency. Thereafter, they recognised the lower quartile Q_1 and upper quartile Q_3 in (i) as the median of the lower half of grouped data and upper half of the grouped data respectively. Therefore, they plugged $N = 50$, $L = 40.5$, $f_b = 36$, $n_w = 17$ and $c = 5$ into the formula $Q_1 = L + \frac{\left(\frac{N}{4} - f_b\right)}{n_w} \times C$ to get the lower quartile Q_1 correct to two decimal places as 41.53. Similarly, they calculated the upper quartile Q_3 to obtain 51.33 by using the formula $Q_3 = L + \frac{\left(\frac{3N}{4} - f_b\right)}{n_w} c$ where by N=50, L=50.5, f_b=36, $n_w = 9$ and $c = 5$. Finally in (ii), they computed the 70[th] percentile correct to one decimal place as 50.0 by using the formula $P_{70} = L + \left(\frac{\frac{7}{10}N - n_b}{n_w}\right) c$, where $n_b = 26$, $n_w = 10$ and $L = 45.5$. Extract 4.1 is a sample of a correct response from one of the candidates who demonstrated skills on manipulating different formulae in answering this question.

4(a) $\sigma_{new}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(2x_i + 1 + -\left(\frac{2(x_1+x_2+x_3 \cdots + x_n)}{n} + 1\right)\right)^2$

$\sigma_{new}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(2x_i + 1 - 2\bar{x} - 1\right)^2$

$\sigma_{new}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(2x_i - 2\bar{x}\right)^2$

$\sigma_{new}^2 = \frac{1}{n} 2^2 \left(\sum_{i=1}^{n}(x_i - \bar{x})\right)$

but $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}) = 100.$

$$\delta new^2 = 2^2 \times 100$$

$$\delta new = \sqrt{4 \times 100}$$

$$\delta new = 20$$

$\therefore$ The standard deviation of $x_1, x_2, x_3 \ldots x_n$ is 10 and $2x_1 + 1, 2x_2 + 1, 2x_n + 1$ is 20

4(b) FREQUENCY DISTRIBUTION TABLE

Length	f	Cum.f	Real limits	X
31 – 35	3	3	30.5 – 35.5	33
36 – 40	6	9	35.5 – 40.5	38
41 – 45	17	26	40.5 – 45.5	43
46 – 50	10	36	45.5 – 50.5	48
51 – 55	9	45	50.5 – 55.5	53
56 – 60	5	50	55.5 – 60.5	58
	N=50			

4(b) (ii)

Position of 70^{th} percentile

$$= \left(\frac{nN}{100}\right)^{th}$$

$$= \left(\frac{70 \times 50}{100}\right)^{th}$$

$$= 35$$

Percentile class $\Rightarrow$ 46 – 50, $L = 45.5$

$$P_n = L_n + \left(\frac{\frac{nN}{100} - fb}{fw}\right)c$$

$fb = 26 \quad fw = 10$

$$P_{70^{th}} = 45.5 + \left(\frac{35 - 26}{10}\right) \times 5$$

$P_{70^{th}} = 50.0$ correct to one decimal place

Extract 4.1: *A correct response to Question 4*

In Extract 4.1, the candidate followed all the necessary steps to calculate the required standard deviation in part (a). He/she had the basic knowledge and skills to calculate the quartiles and percentiles in part (b).

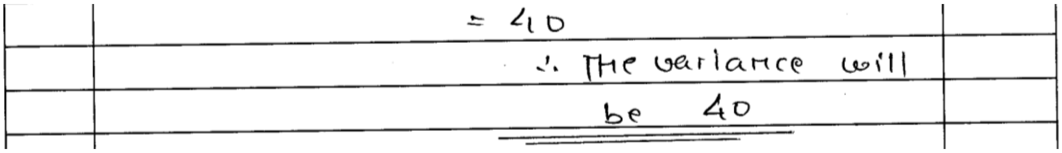
On the other hand, the candidates who answered part (a) incorrectly calculated the new standard deviation using the wrong formula such as $\delta' = \sqrt{\frac{x_1 + x_2 + ... + x_n}{n}}$,

$\delta' = \sqrt{\frac{\sum(2x_n + 1)^2}{n} - \frac{\sum(2\bar{x} + 1)}{n}}$. instead of the correct one

$$\delta' = \sqrt{\frac{\sum_{i=1}^{n}((2x_i + 1) - (2\bar{x} + 1))^2}{n}}.$$

In part (b), some of these candidates failed to decide the classes which contain the first quartile, third quartiles and 70 percentile. Due to this reason, they used incorrect values of f_b, n_w and *L*. One candidate for example, calculated the third quartile Q_3 by using the incorrect class of 36 - 40 instead of 41 - 45. Others were able to select the appropriate classes but did not subtract 0.5 from 41, 51 and 46 to get lower limits for the quartiles and 70th percentile. Such candidates ended up with incorrect lower and upper quartiles like 42.03 and 51.83 and P_{70} equal to 50.5. Furthermore, there were candidates who substituted the midpoint of the class containing the quartiles and percentile. Such candidates got 44.02, 53.25 and 52.5 respectively. Extract 4.2 is a sample response from one of the candidates who had such weaknesses in attempting part (a) of this question.

Q4.	(a) soln	
	From	
	Var(ax+c) = a² Var(x)	
	thus	
	If var(x) = 10	
	then	
	Var(2xn+1) = 4 Var(xn)	
	= 4(10)	



Extract 4.2: *A sample of an incorrect responses to Question 4 (a)*

In Extract 4.2, the candidate substituted an incorrect value of the variance that is $\text{var}(x) = 10$. He/she was supposed to square the given standard deviation so that its respective variance is 100.

2.1.5 Question 5: Sets

The question consisted of parts (a), (b) and (c). In part (a), the candidates were required to simplify the expression $(A \cap B') \cup (A' \cap B) \cup (A \cap B)$ by using the appropriate laws of sets. In part (b), the candidates were informed that in an investigation of eating habits of 110 rabbits; 50 rabbits eat rice, 43 eat maize, 45 eat banana, 12 eat rice and maize, 13 eat maize and banana, 15 eat banana and rice and 5 eat all three types of food. The candidates were required to summarize the given information on a Venn diagram. In part (c), the candidates were required to use the Venn diagram obtained in part (b) to find the number of rabbits which eat: (i) only one type of food; (ii) banana and maize but not rice and; (iii) none of the food.

The question was attempted by 12,702 (99.5%) candidates whereby 8.8 percent scored 3.0 marks or less, 12.5 percent scored from 3.5 to 5.5 marks and 78.7 percent scored from 6.0 to 10.0 marks. In this question, 11,584 (91.2%) candidates scored more than 3.0 marks. Therefore, question 5 was among the five questions which were well done in this examination. Overall the performance in this question was the highest. Figure 6 shows the general performance of candidates in this question.

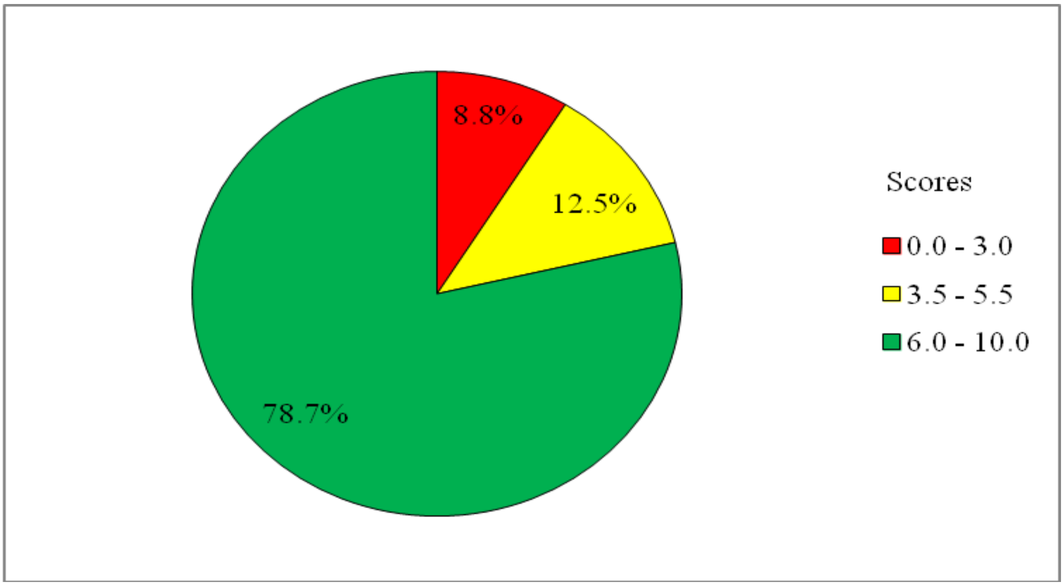
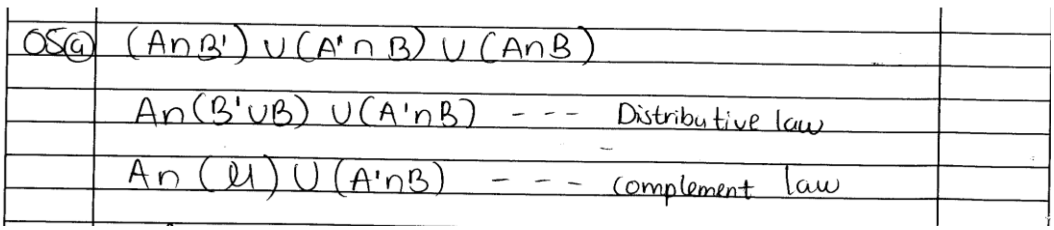


Figure 6: *Candidates' performance in question 5*

The analysis of data shows that a total of 2277 candidates (17.8%) scored all 10 marks. These candidates demonstrated the following competences: In part (a), they were able to apply; the distributive law on the given expression to obtain $(A \cap B') \cup [(A' \cup A) \cap B]$, the compliment law to obtain $(A \cap B') \cup (\mu \cap B)$, the identity law to obtain $(A \cap B') \cup B$, the distributive law to get $(A \cup B) \cap (B' \cup B)$. They were also able to apply the compliment law to get $(A \cup B) \cap \mu$ and finally used identity law to arrive at AÈB as required. In part (b), the candidates presented correctly the given information into a Venn diagram involving three sets. By using the Venn diagram obtained in part (b), these candidates were able to answer correctly the sub-items in part (c) as follows: (i) 73 rabbits eat only one type of food, (ii) 8 eat banana and maize but not rice and (iii) 7 rabbits eat none of the food. Extract 5.1 is a sample of an answer from the script of a candidate who answered the question correctly.


05(a) $(A \cap B') \cup (A' \cap B) \cup (A \cap B)$

$A \cap (B' \cup B) \cup (A' \cap B)$ --- Distributive law

$A \cap (\mu) \cup (A' \cap B)$ --- complement law

$A \cup (A' \cap B)$ - - - Identity law

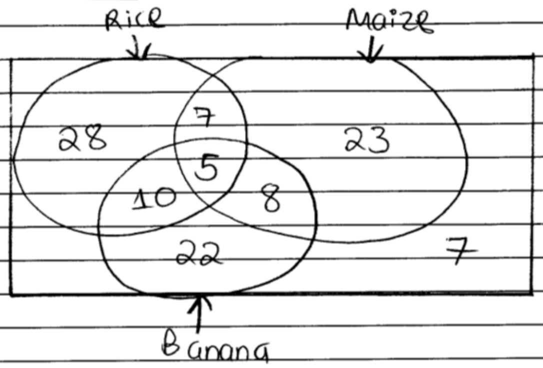
$(A \cup A') \cap (A \cup B)$ - - - Distributive law

$U \cap (A \cup B)$ - - - Complement law

$A \cup B$ - - - Identity law

$= \underline{A \cup B}$

(b)



(c) (i) 73 rabbits

(ii) 8 rabbits

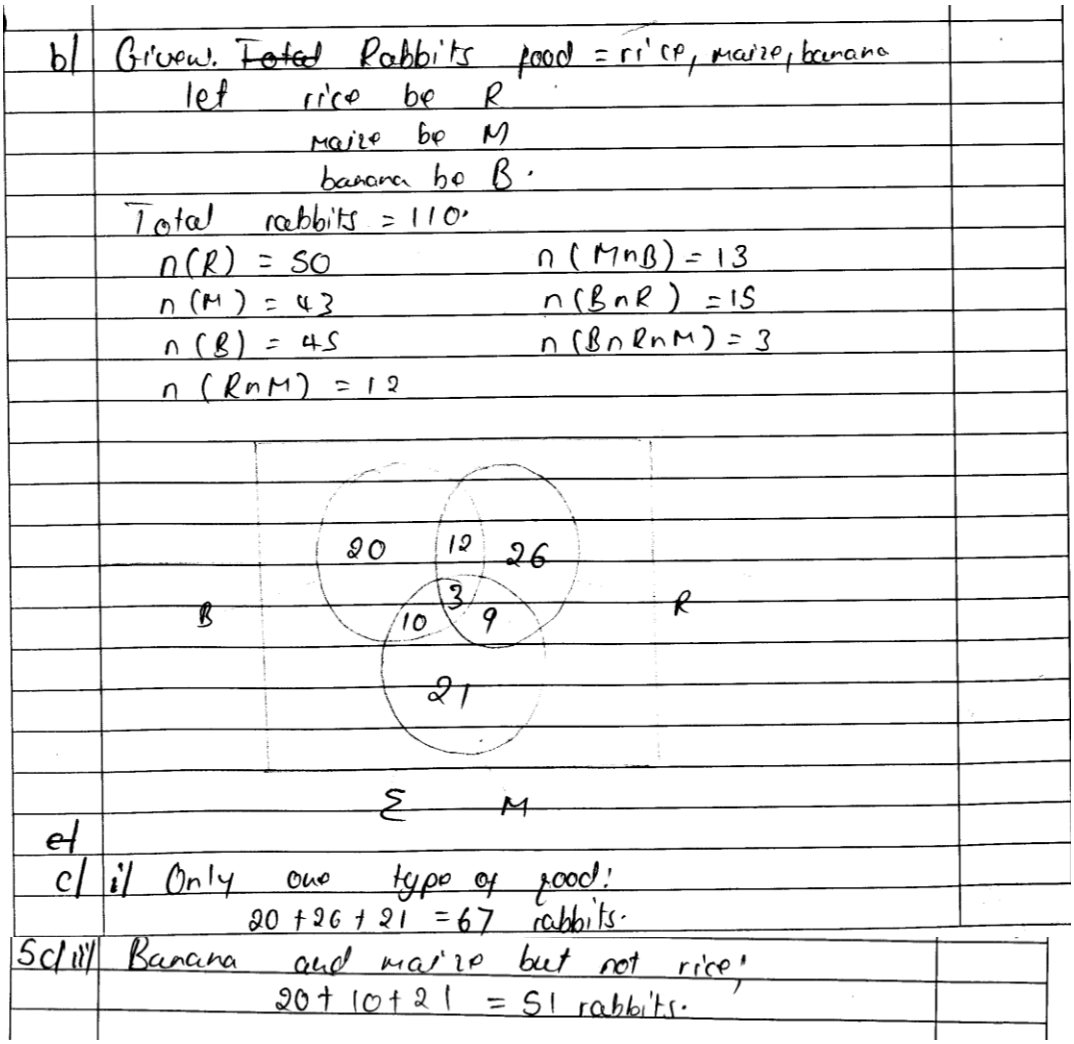
(iii) 7 rabbits

Extract 5.1 *A correct response to Question 5*

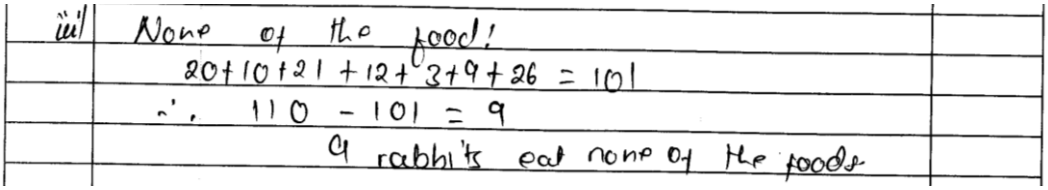
In Extract 5.1, the candidate was able to apply the laws of algebra of sets to simplify the given expression. The candidate also presented the given word problem in a Venn diagram and finally used it to find the number of rabbits under investigation eating different types of foods.

In spite of the candidates' good performance, there were 95 (0.7%) who got the question wrong. The analysis shows that in attempting this question the candidates encountered some challenges as follows: In part (a), some candidates wrote incorrect laws in front of each step. One candidate for instance, changed the given set expression into $(A \cap B') \cup (A' \cup A) \cap (B \cup B)$ by using the distributive law. Other candidates did not write the laws used although their working was correct. Several of these candidates began by expressing $(A \cap B') \cup (A' \cap B) \cup (A \cap B)$

using a wrong set expression like $(A \cup A') \cap (B \cup B') \cap (A \cap B)$. These candidates simplified this expression to get $A \cap B$ or m instead of *A*È*B*. In part (b), they failed to enclose the overlapping circle with a box defining the universal set. Some of the candidates were able to a draw Venn diagram but did not position the number of rabbits correctly in each region of the Venn diagram. In part (c), several candidates encountered challenges in answering the sub-items; (i), (ii) and (iii). There were for instance candidates who added 23, 8 and 22 to get 53 as the number of rabbits which eat banana and maize but not rice. A sample of an incorrect answer from one of the candidates who gave an incorrect answer in part (b) of the question is shown in Extract 5.2.


b/ Given. ~~Total~~ Rabbits food = rice, maize, banana
let rice be R
maize be M
banana be B.
Total rabbits = 110.
n(R) = 50 n(M∩B) = 13
n(M) = 43 n(B∩R) = 15
n(B) = 45 n(B∩R∩M) = 3
n(R∩M) = 12

e/
c/ i/ Only one type of food:
20 + 26 + 21 = 67 rabbits.
5c/ ii/ Banana and maize but not rice,
20 + 10 + 21 = 51 rabbits.


iii) None of the food!
20+10+21+12+3+9+26 = 101
∴ 110 − 101 = 9
9 rabbits eat none of the foods

Extract 5.2: *An incorrect response to Question 5*

In Extract 5.2, the candidate wrote an incorrect quantity of rabbits in each disjoint region. Hence, he/she produced incorrect answers to the sub-questions in part (c).

2.1.6 Question 6: Functions

The question had parts (a), (b) and (c). In part (a), the candidates were required to find $f\left(g\left(x\right)\right)$ given that $f\left(x\right)=\sqrt{x^2-9}$ and $g\left(x\right)=x-2$. In part (b), they were required to draw the graph of $f\left(g\left(x\right)\right)$ obtained in part (a). In part (c), they were given that $y=\frac{x^2-9}{x-1}$ and were required to: (i) find the asymptotes of y and (ii) draw the graph of y.

This question was attempted by 12,743 (99.8%) candidates. The analysis of data shows that 1,520 (12%) candidates scored 3.0 marks or less and 11,223 candidates (88%) scored more than 3.0 marks. This indicates that the candidates' performance in the question was good. Figure 7 summarizes the candidates' performance in this question.

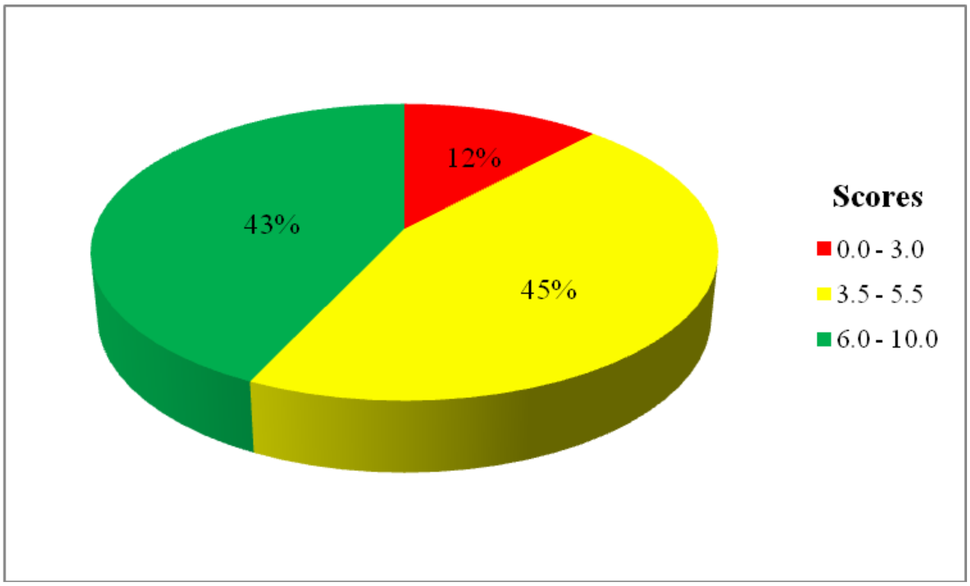


Figure 7: *Candidates' performance in question 6*

Further analysis shows that 5477 candidates (43.0%) scored from 6.0 to 10.0 marks. These candidates substituted $g(x) = x - 2$ into $f(x) = \sqrt{x^2 - 9}$ to get the composite function $f(g(x)) = \sqrt{x^2 - 4x - 5}$ in part (a). In part (b), the candidates correctly solved $f(g(x))$ to get the x - intercepts (-1, 0) and (5, 0), prepared a table of values for $x \le -1$ and x^{3} 5 and correctly traced the path of $f(g(x)) = \sqrt{x^2 - 4x - 5}$. In part (c) (i), the candidates noted that since the degree of the numerator is greater than the degree of the denominator, they performed long division method to obtain $y = (x+1) - \frac{8}{x-1}$ which gave the oblique asymptote $y = x + 1$. Furthermore, they equated the denominator $x - 1$ to zero so as to get the vertical asymptote $x = 1$. In (ii), the candidates solved $0 = \frac{x^2 - 9}{x - 1}$ to obtain x-intercepts at the points $(-3, 0)$ and $(3, 0)$. Thereafter, they substituted $x = 0$ into $y = \frac{x^2 - 9}{x - 1}$ to obtain y-intercept at the point (0, 9). Finally, they traced the path of $y = \frac{x^2 - 9}{x - 1}$ in two regions as shown in Extract 6.1.

6 (a) soln

Given $f(x) = \sqrt{x^2 - 9}$.

$g(x) = x - 2$.

Now

$f(g(x)) = ?$

$f(g(x)) = \sqrt{(x-2)^2 - 9}$.

$f(g(x)) = \sqrt{x^2 - 4x - 5}$.

$\therefore f(g(x)) = \sqrt{x^2 - 4x - 5}$.

(b) Graph of $f(g(x))$.

vertical asymptote

$x^2 - 4x - 5 = 0$

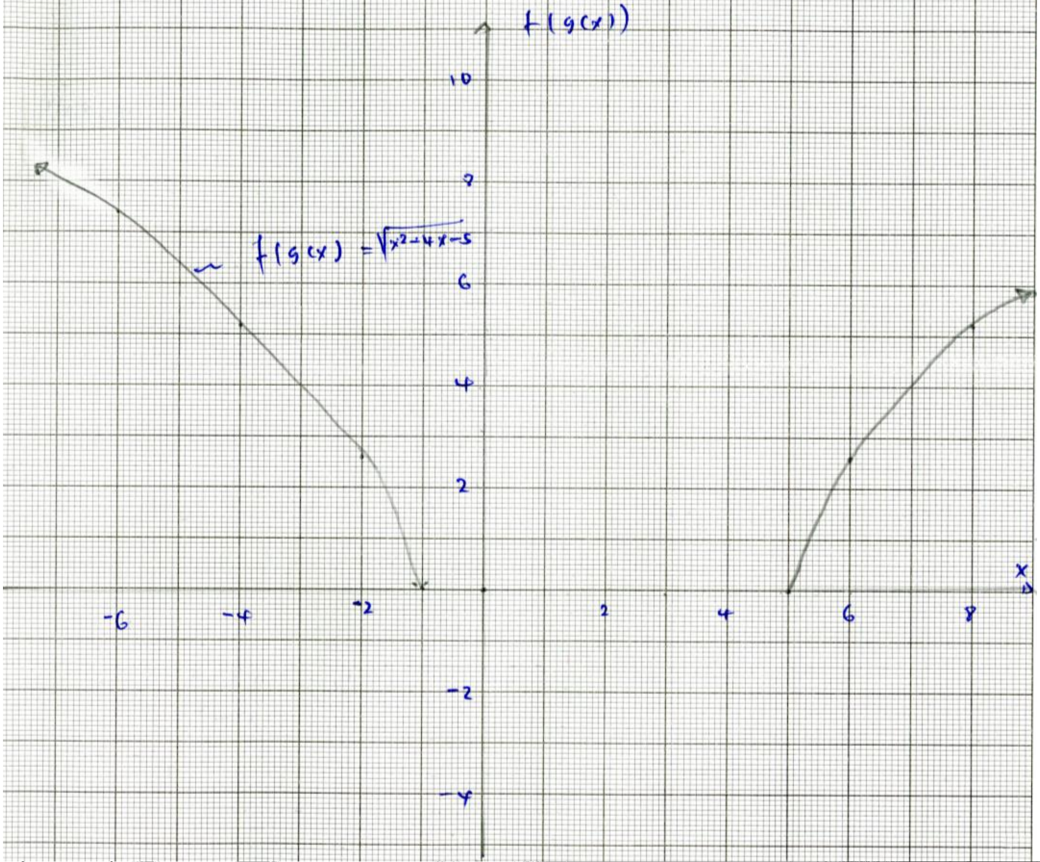
x- intercept $x = 5$ and $x = -1$

(5, 0) and (−1, 0)

Range = {y : y ∈ ℝ ; y ≥ 0}

Domain = {x : x ∈ ℝ ; except −1 < x < 5}.

6 (b) GRAPH of f(g(x)).



6	(c)	soln
		Given
		$y = \frac{x^2-9}{x-1}$.
		(i)
		Vertical asymptote;
		$x - 1 = 0$
		$x = 1$
		slant asymptote;
		$x - 1 \overline{) x^2 - 9}$ quotient $x + 1$
		$x^2 - x$
		$x + 9$
		$x - 1$
		-8

$$y = x + 1 - \frac{8}{x-1}.$$

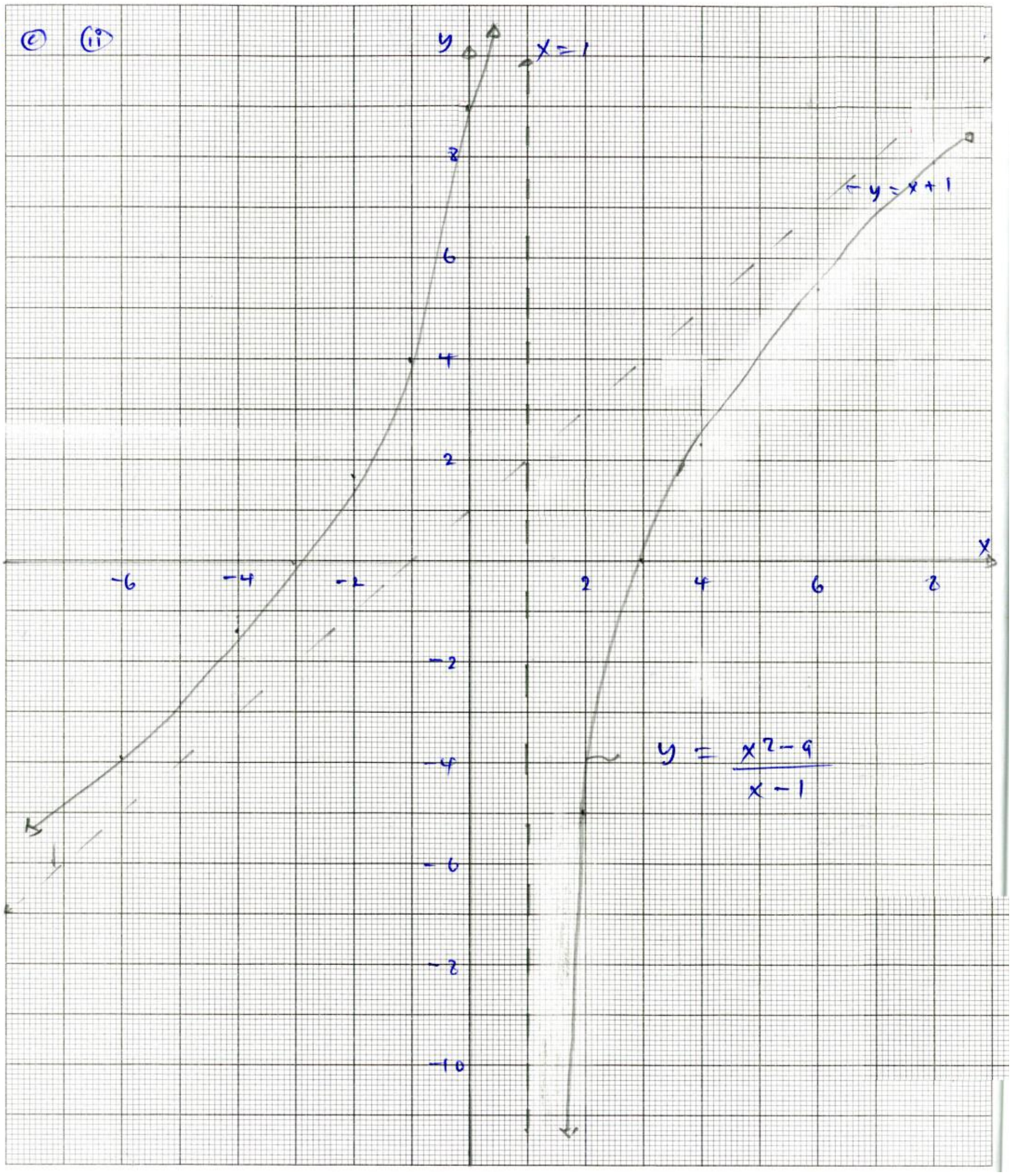
$$x \to \infty$$

$\therefore$ slant asymptote $y = x + 1$.

(ii) Graph of y.

Domain: $\{x : x \in \mathbb{R};$ except $x = 1\}$

Range: {All real number}.

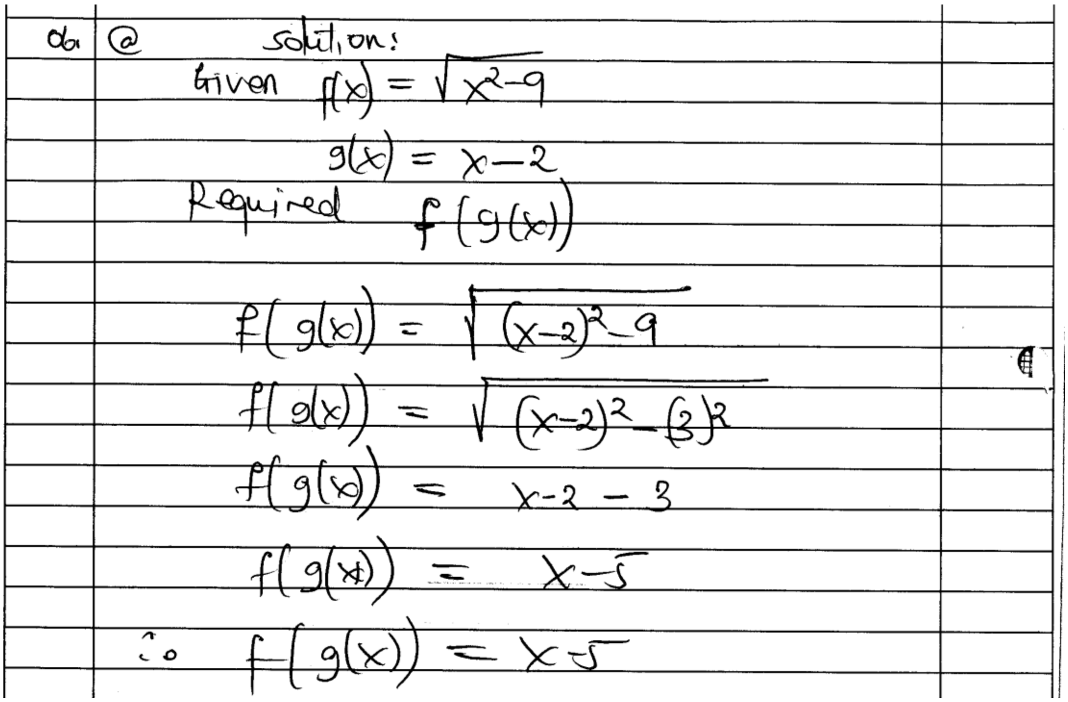


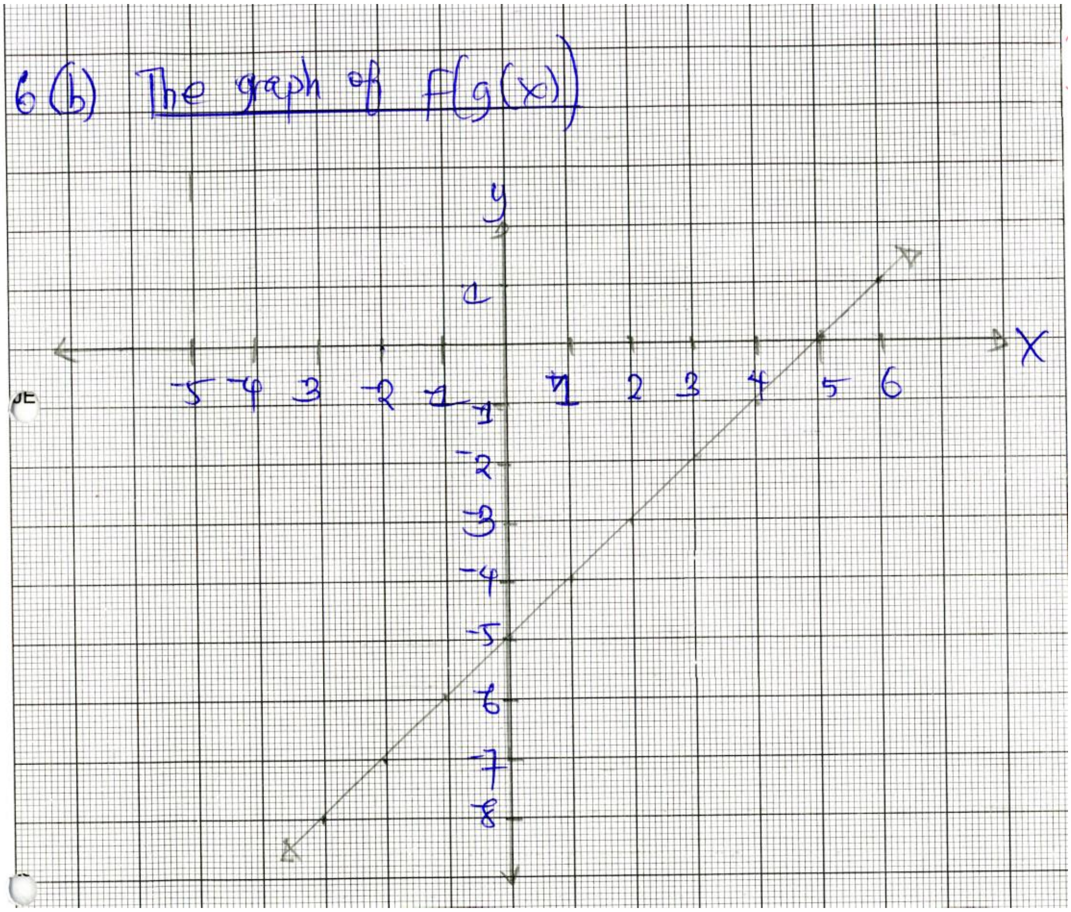
Extract 6.1: *A sample of correct response to Question 6*

In Extract 6.1, the candidate was able to find and draw the graph of the composite function $f\left(g\left(x\right)\right)=\sqrt{x^2-4x-5}$, managed to find the asymptotes of $y=\dfrac{x^2-9}{x-1}$. Finally the candidate demonstrated adequate knowledge and skills on drawing the graph of $y=\dfrac{x^2-9}{x-1}$.

Despite the good performance in this question, there were 1,520 candidates (11.9%) who scored low marks in this question. In part (a), the candidates were not able to find the composite function $f\left(g\left(x\right)\right)$. Some of them conceptualized $f\left(g\left(x\right)\right)$ as $f\left(g\left(x\right)\right)=g(x)\times f(x)$ as they wrote $f\left(g\left(x\right)\right)=\left(x-2\right)\sqrt{x^2-9}$ which is incorrect. Others did not apply $f\left(x\right)$ and $g\left(x\right)$ in the correct order because they quoted $f\left(g\left(x\right)\right)$ as $\sqrt{x^2-9}-5$ which is the composite function of $g\left(f\left(x\right)\right)$. Also, a significant number of these candidates in the process committed some basic algebraic mistakes ending up with incorrect functions like $f\left(g\left(x\right)\right)=\sqrt{x^2-4x-13}$. Another common mistake noted from these candidates was ignoring the presence of the radical sign in the function $f\left(x\right)$ which produced the incorrect composite functions such as $f\left(g\left(x\right)\right)=x^2-4x-5$.

In part (b), some of these candidates failed to draw the required graph due to failure to find correctly $f\left(g\left(x\right)\right)$ in part (a). Others traced the graph of $f\left(g\left(x\right)\right)$ at only one x-intercept (-1,0) which is on the left hand side of the y – axis. These candidates did not understand that $f\left(g\left(x\right)\right)$ can also be traced on the right hand side of the y axis because its domain is $\{x : x \in \mathbb{R} : \text{except} -1 < x < 5\}$. In part (c), a number of candidates confused the procedures of determining an oblique asymptote with that of a vertical asymptote. Some candidates reversed the calculations of these asymptotes. Hence, they could not trace correctly the path for the graph of $y = \frac{x^2 - 9}{x - 1}$. Other candidates in this category could not find correctly the asymptotes. For example one of the candidates wrote $y = \lim_{x \to \infty} \left(\frac{1 - \frac{9}{x^2}}{\frac{1}{x} - \frac{1}{x^2}} \right)$ and obtained the horizontal asymptote of 0 instead of $y = (x+1) - \lim_{x \to \infty} \left(\frac{\frac{8}{x^2}}{\frac{1}{x} - \frac{1}{x^2}} \right)$ to get an oblique asymptote $y = x + 1$. These candidates ended up with incorrect responses such as no vertical asymptote. Extract 6.2 shows a sample of an incorrect response from a candidate who did badly in part (a) and (b) of this question.





Extract 6.2: *A sample of an incorrect response to Question 6*

In Extract 6.2, the candidate substituted *x* with $g(x) = x - 5$ in the function $f(x)$ correctly but simplified $\sqrt{(x-2)^2 - 9}$ to be $x - 5$ instead of $\sqrt{x^2 - 4x - 5}$ in part (a). Due to this mistake, he/she sketched the graph of a straight line passing through the points $(0, -5)$ and $(5, 0)$ in part (b). This shows the candidate had insufficient knowledge and skills on algebra.

2.1.7 Question 7: Numerical Methods

This question consisted of parts; (a), (b), (c) and (d). In part (a), the candidates were required to use the trapezium rule with 5 ordinates to find an approximate value for $\int_0^1 \frac{2dx}{1+x^2}$ correctly to four decimal places. In part (b), they were required to use the Simpson's rule with 5 ordinates to estimate the value of $\int_0^1 \frac{2dx}{1+x^2}$ correct to four decimal places. In part (c), they were required to find the value of the integral $\int_0^1 \frac{2}{1+x^2}dx$ correct to four decimal places. In part (d), they were required to compare the actual value in part (c) with the approximate values obtained in part (a) and (b).

The question was attempted by 97.2 percent of the candidates, whereby 10.2 percent scored from 0 to 3.0 marks, 20.1 percent scored from 3.5 to 5.5 marks and 69.7 percent scored from 6.0 to 10.0 marks. Generally, the candidates' performance was good, as 89.8 percent scored more than 3.0 marks. Figure 8 is a summary of the candidates' performance in this question.

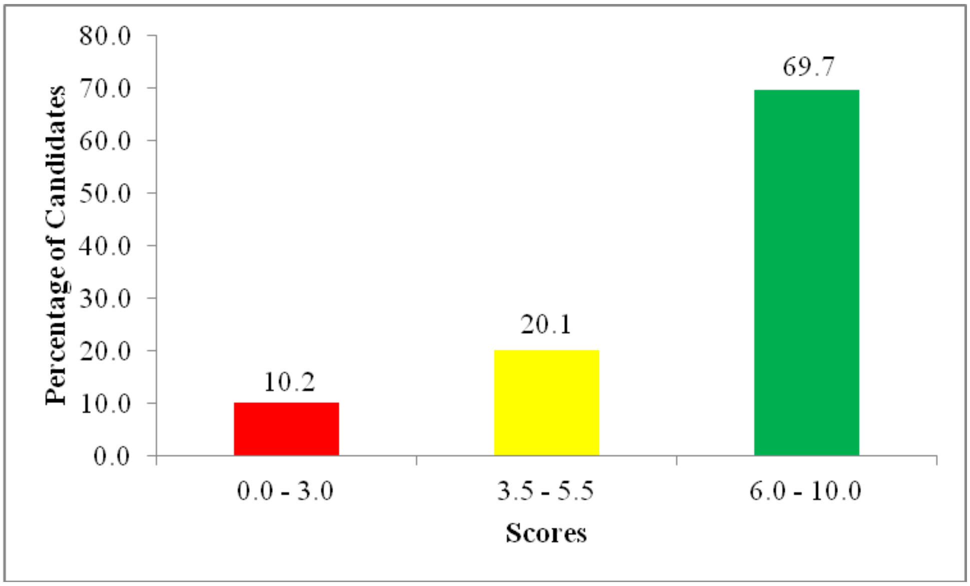


Figure 8: *The candidates' performance in question 8*

The candidates (69.7%) who scored high marks had adequate knowledge of numerical integration. In part (a), they were able to find the value of $h = 0.25$ using the formula $h = \frac{b - a}{n - 1}$ where $a = 0$, $b = 1$ and $n = 5$. They used the value for h to construct a table of values of $y = \frac{2}{1+x^2}$ as shown in Table 3.

Table 3: Table of values

x	$1+x^2$	Ordinates $y=\dfrac{2}{1+x^2}$		
		First and last ordinates	Odd ordinates	Even ordinates
0.00	1.0000	2.0000		
0.25	1.0625		1.8823	
0.50	1.2500			1.6000
0.75	1.5625		1.2800	
1.00	2.0000	1.0000		
		3.0000	3.1623	1.6000

Thereafter, they used table 3 together with the Trapezoidal rule formula $A=\frac{h}{2}\left[y_0+y_4+2(y_1+y_2+y_3)\right]$ to estimate the value of $\int_0^1 \frac{2}{1+x^2}dx$ to four decimal places correctly as 1.5656. In part (b), most of these candidates were familiar with the Simpson's rule. Thus, they substituted ordinates and the width of each strip in the formula $A=\frac{h}{3}\left[y_0+y_4+4\left(y_1+y_3\right)+2y_2\right]$ where $y_0=1.0000$, $y_1=1.0625$, $y_2=1.2500$, $y_3=1.5625$, $y_4=2.0000$ and $h=0.25$ to approximate the value for the integral $\int_0^1 \frac{2}{1+x^2}dx$ as 1.5708 correct to 4 decimal places. In part (c), the candidates used the substitution $x=\tan\theta$ to change the integration variables and limits to $2\int_0^{\frac{\pi}{4}} d\theta$ which was computed to get 1.5708 correct to 4 decimal places. This indicates that the candidates had adequate knowledge and skills on techniques of integration. In part (d), they subtracted the actual value from approximated values of the integral $\int_0^1 \frac{2}{1+x^2}dx$ to compute the absolute error in Trapezium and Simpson's rule as 0.0052 and 0.0000 respectively. Finally, the candidates concluded that Simpson's rule gives a better approximation as it has a smaller absolute error when compared to the Trapezium rule. Extract 7.1 is a sample solution obtained from a candidate who answered question 7 correctly.

7 (a) Given.

$$\int_0^1 \frac{2}{1+x^2}\,dx.$$

For 5 ordinates = 4 stripes

$h = \frac{1-0}{4} = 0.25.$

x	0	0.25	0.5	0.75	1.
$y = \frac{2}{1+x^2}$	2	1.88235	1.6	1.28	1.
Ordinates	y_0	y_1	y_2	y_3	y_n.

From Trapezium Rule.

$$\int_0^1 \frac{2}{1+x^2}\,dx = \frac{h}{2}\left[y_0 + y_n + 2\Sigma \text{all ordinates}\right].$$

$$= \frac{0.25}{2}\left[2 + 1 + 2(1.88235 + 1.6 + 1.28)\right].$$

$= 1.5656.$

(b) $\int_0^1 \frac{2}{1+x^2}\,dx.$

For 5 ordinates, Stripes = 4.

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1.
$y = \frac{2}{1+x^2}$	2	1.88235	1.6	1.28	1.
Ordinates	y_0	y_1	y_2	y_3	y_n.

For Trapezium rule.

$$\int_0^1 \frac{2}{1+x^2}\,dx = \frac{h}{3}\left[y_0 + y_n + 2\Sigma \text{even ordinates} + 4\Sigma \text{odd ordinates}\right]$$

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7 (b) $$\int_0^1 \frac{2}{1+x^2}\,dx = \frac{0.25}{3}\left[2 + 1 + 2(1.6) + 4(1.88235 + 1.28)\right].$$

$= 1.5708$

$$\therefore \int_0^1 \frac{2}{1+x^2}\,dx = 1.5708.$$

ⓒ $\int_0^1 \frac{2}{1+x^2}\,dx.$

$$= 2\int_0^1 \frac{1}{1+x^2}\,dx.$$

let $1+x^2 = 1+\tan^2\theta.$

$$x = \tan\theta.$$

$$\frac{dx}{d\theta} = \sec^2\theta.$$

$$dx = \sec^2\theta\,d\theta.$$

$$= 2\int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{1+\tan^2\theta}$$

$$= 2\int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec^2\theta}$$

$$= 2\theta\Big]_0^{\pi/4}.$$

$$= \frac{\pi}{2} - 0.$$

$$= 1.5708.$$

$$\therefore \int_0^1 \frac{2\,dx}{1+x^2} = 1.5708.$$

7 ⓓ Comparison.

Between the actual value with approximated value from Trapezium rule.

$$\Delta y = |y-y'|$$

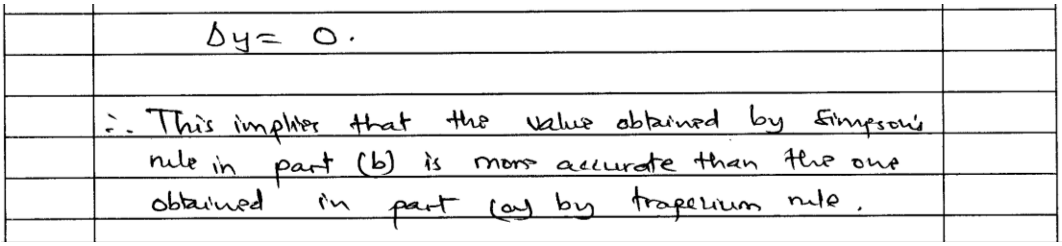
$$= |1.5708 - 1.5656|$$

$$= 0.0052.$$

Between actual value with approximated value from Simpson's rule.

$$\Delta y = |y-y'|$$

$$\Delta y = |1.5708 - 1.5708|.$$



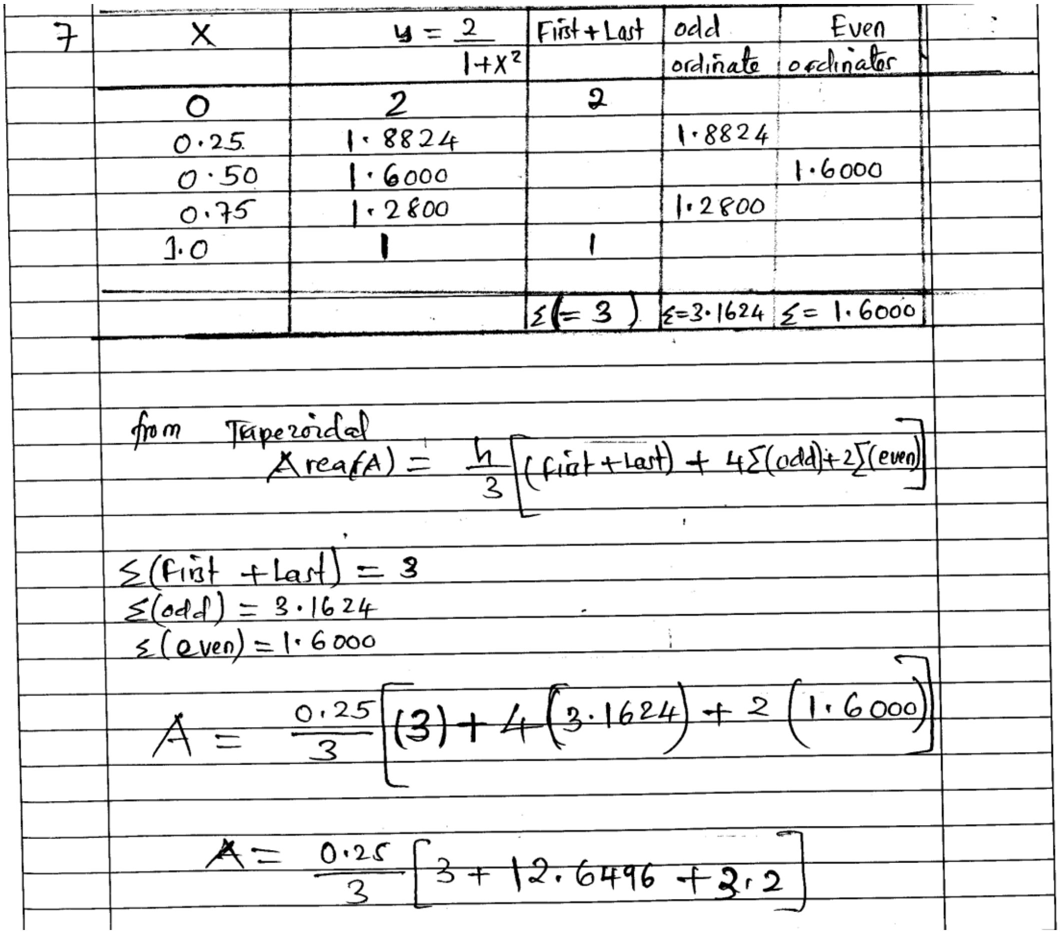
$\Delta y = 0.$

$\therefore$ This implies that the value obtained by Simpson's rule in part (b) is more accurate than the one obtained in part (a) by trapezium rule.

Extract 7.1 *A sample of a correct response to Question 7*

In Extract 7.1, the candidate found correctly the value of h as $h = 0.25$ and prepared the correct table of values. The candidate used both the Trapezium and Simpson's rule formulas and data from the table to approximate the value of $\int_0^1 \frac{2}{1+x^2}dx$. The candidate was also able to find the actual value of $\int_0^1 \frac{2}{1+x^2}dx$ and finally concluded the best method of approximation.

On the other hand, there were 141 candidates (1.1%) who scored 0. The performance of these candidates was weak in this question for various reasons. In part (a) and (b), they computed the width h for each strip using the formula $h = \frac{b-a}{n}$ instead of $h = \frac{b-a}{n-1}$ to get $h = 0.2$ instead of $h = 0.25$. These incorrect values of h produced wrong estimations of the integral $\int_0^1 \frac{2}{1+x^2}dx$ in both Simpson's and Trapezium rules as 1.5675 and 1.4374 respectively. In addition, a number of candidates used the Trapezium rule instead of Simpson's rule formula. These candidates wrote incorrect conclusions such as the estimation obtained by Trapezium rule is more accurate because it is equal to the actual value for the integral $\int_0^1 \frac{2}{1+x^2}dx$. In part (c), most candidates used a scientific calculator to obtain the value of $\int_0^1 \frac{2}{1+x^2}dx$ as 1.5708 correct to four decimal places instead of using an analytical approach. There were also candidates who tried to solve the integral $\int_0^1 \frac{2}{1+x^2}dx$ algebraically, but they had inadequate knowledge and skills of applying techniques of integration. Such candidates were able to convert the integral $\int_0^1 \frac{2}{1+x^2}dx$ into $\int_0^1 \frac{\sec^2\theta}{1+\tan^2\theta}d\theta$ but did not change the limits of integration. It was further noted that, some of these candidates chose $u = x^2 + 1$ instead of $x = \tan\theta$. They ended up getting $\int_0^1 \frac{du}{xu}$ due to this reason. Moreover, several candidates gave the result for $\int \frac{dx}{1+x^2}$ as $\tan^{'} x$ without showing the necessary steps to arrive at the answer. In part (d), the candidates failed to find the absolute error for both Simpson's and Trapezium rules in order to draw the conclusion for the best method of approximation because they committed mistake in part (a) and (b). Extract 7.2 shows a sample of an incorrect response from such candidates.



7	X	$y = \frac{2}{1+x^2}$	First + Last	odd ordinate	Even ordinator
	0	2	2		
	0.25	1.8824		1.8824	
	0.50	1.6000			1.6000
	0.75	1.2800		1.2800	
	1.0	1	1		
			Σ = 3	Σ=3.1624	Σ= 1.6000

from Tapezoidal
Area(A) = $\frac{h}{3}\left[(\text{first}+\text{Last}) + 4\Sigma(\text{odd}) + 2\Sigma(\text{even})\right]$

Σ(First + Last) = 3
Σ(odd) = 3.1624
Σ(even) = 1.6000

$A = \frac{0.25}{3}\left[(3) + 4(3.1624) + 2(1.6000)\right]$

$A = \frac{0.25}{3}\left[3 + 12.6496 + 3.2\right]$

$$A = \frac{0.25}{3}(18.8496)$$

Area of trapezoidal $= 1.5708$

7b) Use simpson's rule.

$n = 5 - 1$

$$h = \frac{b-a}{n} \qquad y = \frac{2}{1+x^2}$$

$$h = \frac{1-0}{4} = 0.25$$

$$\text{Area} = \frac{h}{2}\left[(\text{first} + \text{last}) + 2\Sigma(\text{middle ordinates})\right]$$

x	$y = \frac{2}{1+x^2}$	First + Last	middle ordinates
0	2	2	
0.25	1.8824		1.8824
0.50	1.6000		1.6000
0.75	1.2800		1.2800
1	1	1	
		$\Sigma(\text{first} + \text{last}) = 3$	$\Sigma = 4.7624$

$$A = \frac{0.25}{2}\left[(3) + 2(4.7624)\right]$$

$$A = \frac{0.25}{2}(12.5248)$$

$A = 1.5656$.

$\therefore$ Area by using Simpson's rule $= 1.5656$

Extract 7.2: *A sample of an incorrect response to Question 7*

In Extract 7.2, the candidate was unable to estimate the value of $\int_0^1 \frac{2}{1+x^2}dx$ as he/she confused the formula of Trapezium with that of Simpson's rule.

2.1.8 Question 8: Coordinate Geometry I

The question had parts (a) and (b). In part (a) (i), the candidates were informed "the point R(x, y) divides a line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio *m*:*n* internally and they were required to derive the formula for the ratio theorem. In part (a) (ii), they were required to use the ratio formula obtained in (i) to find the coordinates of the point which divides the line segment joining the points (5,-4) and (-3,2) internally in the ratio 1:2.

In part (b) the candidates were required to (i) derive the formula for the area of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ and (ii) use the formula obtained in (i) to find the area of a triangle with the vertices $A(3,1)$, $B(2k,3k)$ and $C(k,2k)$, and then, show that the vertices are collinear when $k = -2$.

The analysis of data shows that 11,493 candidates (90.0%) attempted this question, whereby 34.3 percent scored 0 to 3.0 marks, 25.0 percent scored from 3.5 to 5.5 marks and 40.7 percent scored 6.0 marks and above. The candidates' performance in this question was good because 65.7 percent of the candidates scored more than 3.0 marks. Figure 9 is a summary of candidates' performance in this question.

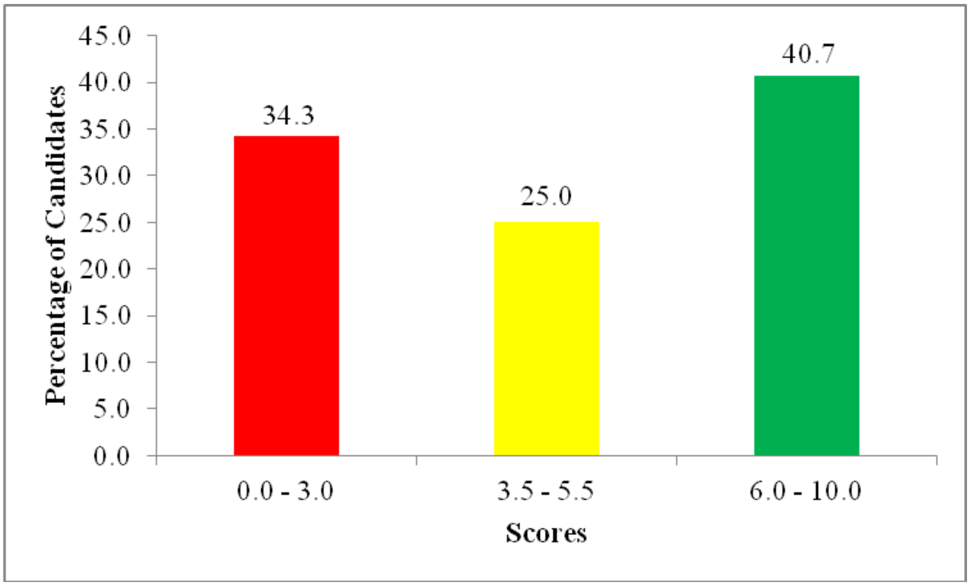
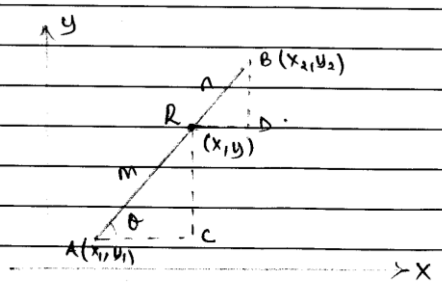


Figure 9: *Candidates' performance in question 8*

The candidates with good performance demonstrated the following competences: The candidates who answered part (a) (i) correctly were able to draw a suitable figure that was used as a guide to establish the ratio theorem. From this figure, they used the concepts of similar triangle to derive the ratio theorem which is $R(x, y) = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$ where (x_1, y_1) and (x_2, y_2) are the end points of a line segment to be divided internally in the ratio $m:n$. In part (a) (ii), the candidates were able to identify the value of *m* and *n* as 1 and 2 respectively and to substitute the coordinates of the points (5,-4) and (-3,2) together with the values of *m* and *n* in the ratio theorem formula to obtain $x = \frac{7}{3}$ and $y = -2$.

In part (b) (i), many candidates arrived at the correct answer by drawing a suitable triangle from which, they were able to use the concepts for area of trapezium to derive the area of a triangle as $\frac{1}{2}\left[(x_1y_2 - x_1y_3) + (x_2y_3 - x_2y_1) + (x_3y_1 - x_3y_2)\right]$. In part (b) (ii), the candidates used the formula obtained in (i) and the coordinates of the points A(3,1), B(2k,3k), C(k,2k) to get the area $A = \frac{1}{2}\left|(k^2 + 2k)\right|$ and they replaced k with -2 in the formula $A = \frac{1}{2}\left|(k^2 + 2k)\right|$ to get $A = 0$. Finally, they wrote a conclusion which says points A, B and C are collinear because the area of the triangle is zero. Extract 8.1 represents a sample solution from one of the candidates who answered this question correctly.

8. a. i.



from the graph.

$\sin\theta = \frac{CR}{AR} = \frac{DB}{BR}$.

8. a. i. but $\frac{AR}{BR} = \frac{m}{n}$.

$\sin\theta = \frac{y - y_1}{m} = \frac{y_2 - y}{n}$.

$ny - ny_1 = my_2 - my$.

$my + ny = my_2 + ny_1$

$y(m+n) = my_2 + ny_1$.

$y = \frac{my_2 + ny_1}{m+n}$.

from the graph.

$\cos\theta = \frac{AC}{AR} = \frac{DR}{BR}$.

$\frac{x - x_1}{m} = \frac{x_2 - x}{n}$.

$nx - nx_1 = mx_2 - mx$.

$mx + nx = mx_2 + nx_1$

$x(m+n) = mx_2 + nx_1$.

$x = \frac{mx_2 + nx_1}{m+n}$

∴ The ratio theorem.

$$R(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

ii. let the point be (x, y)

from the ratio theorem.

$$x = \frac{mx_2 + nx_1}{m+n}.$$

$$= \frac{1(-3) + 2(5)}{1+2}.$$

$$= \frac{-3+10}{3}.$$

use only

8. a. ii. $x = 7/3$

$$y = \frac{my_2 + ny_1}{m+n}$$

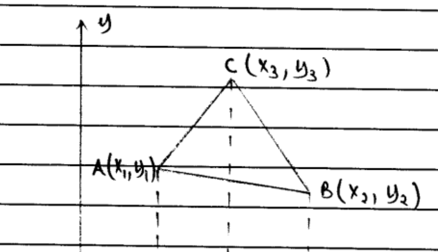
$$= \frac{1(2) + 2(-4)}{1+2}.$$

$$= \frac{2-8}{3}$$

$$= -2$$

∴ Point $(x, y) = (7/3, -2)$.

b. i.



Area of ΔABC = area of ABCDEF − area of ABDEF.

area of ABCDEF = area of ACED + area of BCEF

$$= \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) + \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3).$$

area of ABDEF $= \tfrac{1}{2}(y_2 + y_1)(x_2 - x_1).$

area of $\Delta ABC = \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) + \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3) - \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1)$

$$= \tfrac{1}{2}\left[y_1x_3 - \cancel{y_1x_1} + \cancel{y_3x_3} - y_3x_1 + \cancel{y_2x_2} - y_2x_3 + y_3x_2 - \cancel{y_3x_3} - y_1x_2 + \cancel{y_1x_1} - \cancel{y_2x_2} + y_2x_1\right]$$

$= \frac{1}{2}(y_1x_3 - y_3x_1 - y_2x_3 + y_3x_2 - y_1x_2 + y_2x_1)$.

$= \frac{1}{2}(y_2x_1 - y_3x_1 + y_3x_2 - y_1x_2 + y_1x_3 - y_2x_3)$

$= \frac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]$

$= \frac{1}{2}[3(3k - 2k) - 2k(1 - 2k) + k(1 - 3k)]$

$= \frac{1}{2}[3(k) - 2k + 4k^2 + k - 3k^2]$.

$= \frac{1}{2}(3k - 2k + k + k^2)$

$= \frac{1}{2}(k^2 + 2k)$.

$\therefore$ Area of triangle $= \frac{1}{2}(k^2 + 2k)$ square units.

For collinearity area of triangle = 0.

area $= \frac{1}{2}(k^2 + 2k)$

when $k = -2$.

area $= \frac{1}{2}[(-2)^2 + 2(-2)]$.

$= \frac{1}{2}(4 - 4)$

$= \frac{1}{2}(0)$

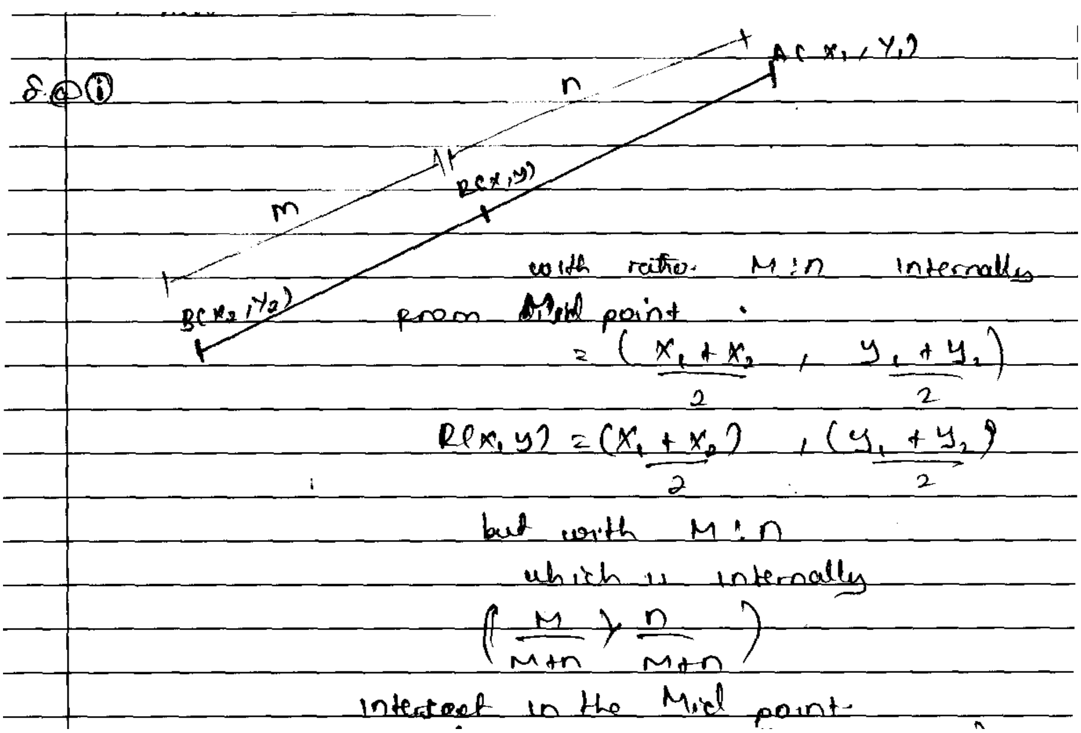
$= 0$.

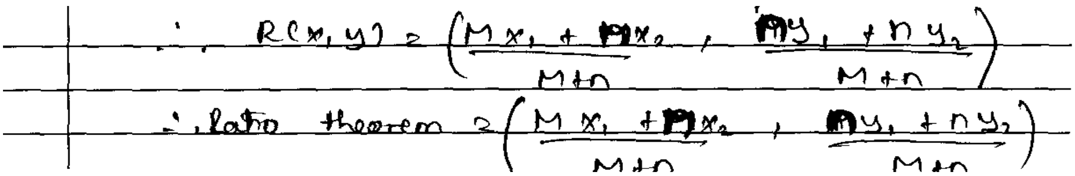
$\therefore$ Since the area of the triangle is 0, then the vertices are collinear when $k = -2$.

Extract 8.1: *A sample of a correct response to Question 8*

In extract 8.1, the candidate had adequate knowledge and skills in deriving and applying the ratio theorem for internal division between two points. The candidate was also able to derive the formula for calculating the area of a triangle using three vertices.

On the other hand, the candidates who got low marks in this question had the following problems: In part (a) (i), a number of these candidates derived the ratio theorem formula for the point which divides a line segment externally in the ratio $m:n$ to get $R(x, y) = \left(\frac{ny_2 - my_1}{m - n}, \frac{nx_2 - mx_1}{m - n}\right)$ contrary to the requirements of the question. They got incorrect coordinates in (ii) due to this reason. There were also candidates who got an incorrect formula because they interchanged the position of *m* and *n*. These candidates ended up getting $R(x, y) = \left(\frac{mx_1 + nx_2}{m + n}, \frac{my_1 + ny_2}{m + n}\right)$ in (i) and the point $\left(-2, \frac{7}{3}\right)$ in (ii). It was also noted that several candidates substituted incorrect coordinates of x and y in the formula as follows $\left(\frac{1(-4)+2(5)}{1+2}, \frac{1(-2)+2(3)}{1+2}\right)$ to obtain $\left(-2, \frac{4}{3}\right)$. In part (b) (i), a number of candidates did not realize that they were supposed to locate the given points on an xy – plane and then use the concept of area to derive the formula to find the area of a triangle indicating failure to comprehend the tested concepts. In part (b) (ii), the candidates managed to find the area of the triangle whose vertices are $A(3,1)$, $B(2k,3k)$ and $C(k,2k)$ but did not show that the points A, B and C are collinear if the area of the triangle is zero. Instead, they equated the area $A=\frac{1}{2}\left|(k^2+2k)\right|$ with zero and then solved the resulting equations to obtain $k=-2$. However, the question did not require them to do so. Extract 8.2 is a sample of an incorrect response from one of the candidates in part (a) of question 8.





Extract 8.2: *An incorrect response to question 8*

Extract 8.2 shows that the candidate had inadequate knowledge and skills in deriving the formula for the ratio theorem.

2.1.9 Question 9: Integration

The question consisted of parts (a) and (b). In part (a), the candidates were required to find $\int x^3 \cos x\, dx$. In part (b), they were required to prove that $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx = \frac{\pi}{4} - \frac{1}{2}$.

Most candidates (66.1%) scored from 0 to 3.0 marks out of 10 marks and a few (5.9%) scored more than 6.0 marks. In addition, 3,241 candidates (32.5%) scored zero because they lacked knowledge and skills on the topic of Integration. Figure 10 shows the percentage of candidates who obtained weak, average and good performance.

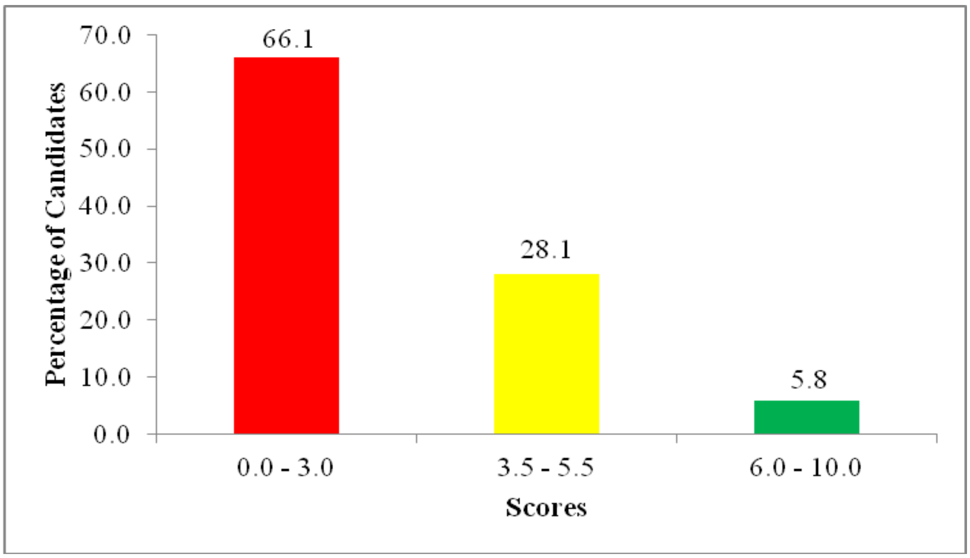
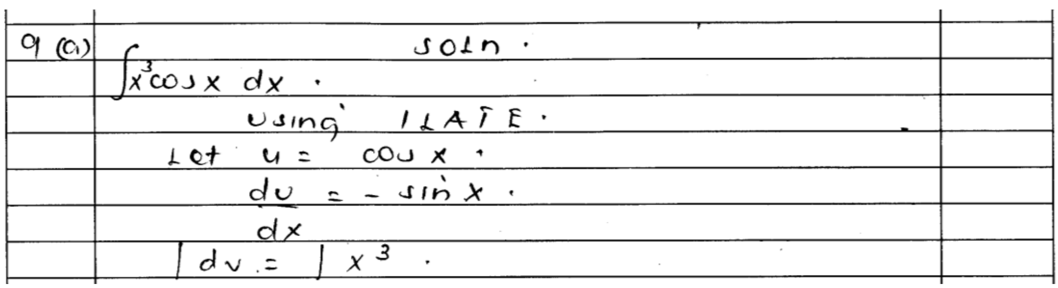
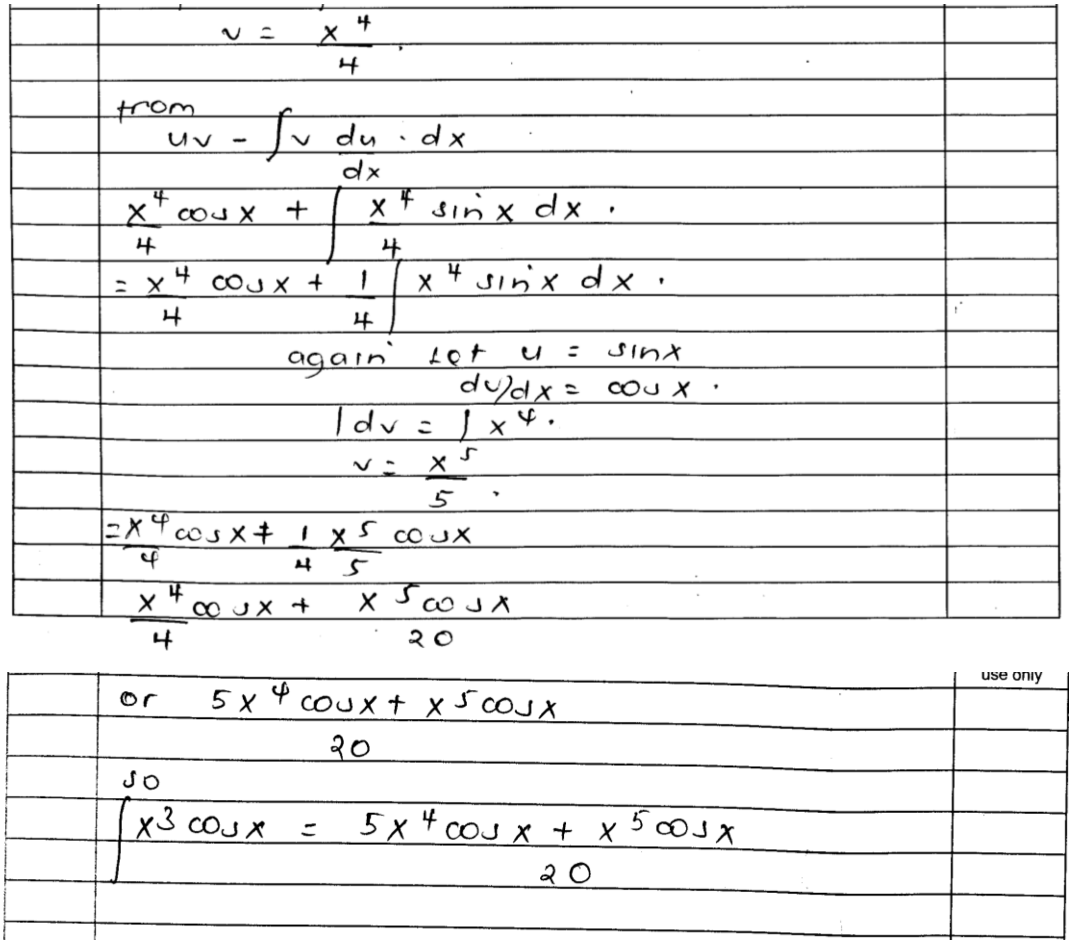


Figure 10: *Candidates' performance in question 9*

The analysis done on the scripts of candidates shows that the candidates with low marks had some weaknesses as follows: In part (a), most candidates integrated $dv = \cos x dx$ to get $v = -\sin x$ instead of $v = \sin x$. These candidates got the incorrect answer $\int x^3 \cos x\, dx = x^2 \cos x - 2\int x \cos x dx$. Other candidates in this category did not consider the mnemonic rule for choosing u and dv that is LIATE where letters L, I, A, T and E stands for Logarithmic, Inverse, Trigonometric and Exponential functions respectively. Such candidates choose $u = \cos x$ instead of $u = x^3$ because x^3 is an algebraic function which is higher in LIATE than the trigonometric function $\cos x$. This resulted in getting the integral $\int x^4 \sin x dx$ which they could not work on. Several candidates used incorrect formulae of integration by parts like $\int u dv = uv + \int v du$ and $\int u dv = u\int dv + v\int du$ instead of $\int u dv = uv - \int v du$. These candidates worked out the given integral which gave incorrect answers like $\int x^3 \cos x\, dx = -\frac{x^4}{4}\cos x + x^3 \sin x + c$.

In part (b), anomalies noted in the scripts of candidates include: failure to choose a suitable substitution to make the given integral easier to evaluate, for example, choosing the substitution $x = \sin q$ instead of $t^2 = 1 + x^2$. Another common mistake was failure of candidates to look at the limits; candidates were unable to change the limits of integration that is when $x = 0 \Rightarrow t = 1$ and $x = 1 \Rightarrow t = \sqrt{2}$. Other candidates were unable to correctly substitute the limits. Due to this reason they were not able to prove the correctness of the given integral. Extract 9.1 is a sample of an incorrect answer from one of the candidates to question 9 (a).


9 (a) soln.
$\int x^3 \cos x\ dx$.
using ILATE.
Let $u = \cos x$.
$\frac{du}{dx} = -\sin x$.
$\int dv = \int x^3$.



$v = \frac{x^4}{4}$

from

$uv - \int v \frac{du}{dx} \cdot dx$

$\frac{x^4}{4}\cos x + \int \frac{x^4}{4}\sin x \, dx$

$= \frac{x^4}{4}\cos x + \frac{1}{4}\int x^4 \sin x \, dx$

again let $u = \sin x$

$du/dx = \cos x$

$\int dv = \int x^4$

$v = \frac{x^5}{5}$

$= \frac{x^4}{4}\cos x + \frac{1}{4}\frac{x^5}{5}\cos x$

$\frac{x^4}{4}\cos x + \frac{x^5 \cos x}{20}$

or $\frac{5x^4\cos x + x^5 \cos x}{20}$

so

$\int x^3 \cos x = \frac{5x^4 \cos x + x^5 \cos x}{20}$

Extract 9.1: *Candidates' incorrect answer to Question 9*

In Extract 9.2, the candidate did not use the LIATE technique as he/she chose the trigonometric function $u = \cos x$.

Despite the candidates' poor performance in this question, there were 281 candidates (2.2%) who scored all ten marks. In part (a), they demonstrated the following competences: Based on the integrand and the preferred LIATE order, they preferred to choose $u = x^3$ and $dv = \cos x dx$; Then they computed du and v to get $du = 3x^2 dx$ and $v = \sin x$ respectively. Finally they put $u = x^3$, $dv = \cos x dx$, $du = 3x^2 dx$ and $v = \sin x$ in the formula for integration by parts $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ to give $\int x^3 \cos x \, dx = x^3 \sin x - 3\int x^2 \sin x \, dx$ which upon simplification produced the correct answer which is $\left(x^3-6x\right)\sin x+\left(3x^2-6\right)\cos x+A$.

In part (b), some candidates with adequate knowledge managed to: choose the substitution $x^2=\cos 2\theta$; find dx in terms of $d\theta$ as $dx=-\frac{\sin 2\theta}{x}$; and to substitute all the new variables in the given definite integral to get $\int_0^{\frac{\pi}{4}} \sin 2\theta\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,d\theta$. Finally they computed the new integral to prove the correctness of $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx=\frac{\pi}{4}-\frac{1}{2}$. Other candidates in this category resorted to select the substitution $t^2=1+x^2$. They also managed to prove that $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx=\frac{\pi}{4}-\frac{1}{2}$. Extract 9.1 shows a correct sample of responses from one of the candidates who had knowledge of the concepts tested.

use only

9 (a)

Soln'

Given'

$$\int x^3 \cos x \, dx.$$

Integration by part (ILATE)

From

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}.$$

$$u = x^3. \Rightarrow \frac{du}{dx} = 3x^2$$

$$\frac{dv}{dx} = \cos x \qquad v = \sin x.$$

Now

$$\int x^3 \cos x \, dx = x^3 \sin x - 3\int x^2 \sin x \, dx.$$

$$= x^3 \sin x - 3\int x^2 \sin x \, dx \quad \text{—(1)}$$

Also for

$$\int x^2 \sin x \, dx = -x^2\cos x + 2\int x\cos x \, dx.$$

$$\int x\cos x \, dx = x\sin x - \int \sin x \, dx$$

$$= x\sin x + \cos x + c.$$

Then

$$\int x^3\cos x \, dx = x^3\sin x - 3\left[-x^2\cos x + 2(x\sin x + \cos x)\right]$$

$$\therefore \int x^3\cos x \, dx = x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + c$$

9 (b) soln

To prove

$$\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}} \, dx = \frac{\pi}{4} - \frac{1}{2}$$

consider L.H.S

$$= \int_0^1 \frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}} \, dx.$$

let $x^2 = \cos 2\theta$

$$2x \, dx = -2\sin 2\theta \, d\theta$$

$$x \, dx = -\sin 2\theta \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \; -\sin 2\theta \, d\theta.$$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{\cos^2\theta + \sin^2\theta - \cos^2\theta + \sin^2\theta}{\cos^2\theta + \sin^2\theta + \cos^2\theta - \sin^2\theta}} \; (-\sin 2\theta) \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \cdot (-\sin 2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{0} \frac{\sin\theta}{\cos\theta} \cdot (-\sin 2\theta)\, d\theta$$

(b)

$$= \int_{\frac{\pi}{4}}^{0} \frac{\sin\theta}{\cos\theta} (-2\sin\theta\cos\theta)\, d\theta$$

$$-\int_a^b f(x)\,dx = \int_b^a f(x)\,dx.$$

$$= \int_0^{\pi/4} 2\sin^2\theta\, d\theta.$$

but $2\sin^2\theta = 1 - \cos 2\theta$

$$= \int_0^{\pi/4} (1 - \cos 2\theta)\, d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\right] - \left[0 - 0\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx = \frac{\pi}{4} - \frac{1}{2}$$

Hence proved

Extract 9.1: *A Candidate's correct response to Question 9*

In Extract 9.1, the candidate used LIATE in choosing the correct expressions of u and dv in part (a). In part (b), the candidate was able to choose a correct substitution to make the given integrand easier to compute.

2.1.10 Question 10: Differentiation

The question comprised parts (a) and (b). In part (a), the candidates were required to show that there is a solution to the equation $x^3 - 6x^2 + 9x + 1 = 0$ between $x = -1$ and $x = 0$. Candidates were required to sketch the curve given by $y = x^3 - 6x^2 + 9x + 1$ without using the table of values. In part (b), they were required to use Taylor's theorem to expand $(x+h)^{\frac{1}{2}}$ in ascending powers of h up to the term containing h^3. Candidates were also required to obtain the value of $\sqrt{10}$ correct to five decimal places.

This question was attempted by 8822 (69.1%) candidates whereby only 7.5 percent scored above 6.0 marks and 17.7 percent scored from 3.5 to 5.5 marks. It was the question that had the lowest performance in this examination because 74.8 percent scored not more than 3.0 out of 10 marks and 14.9 percent of these candidates scored zero.

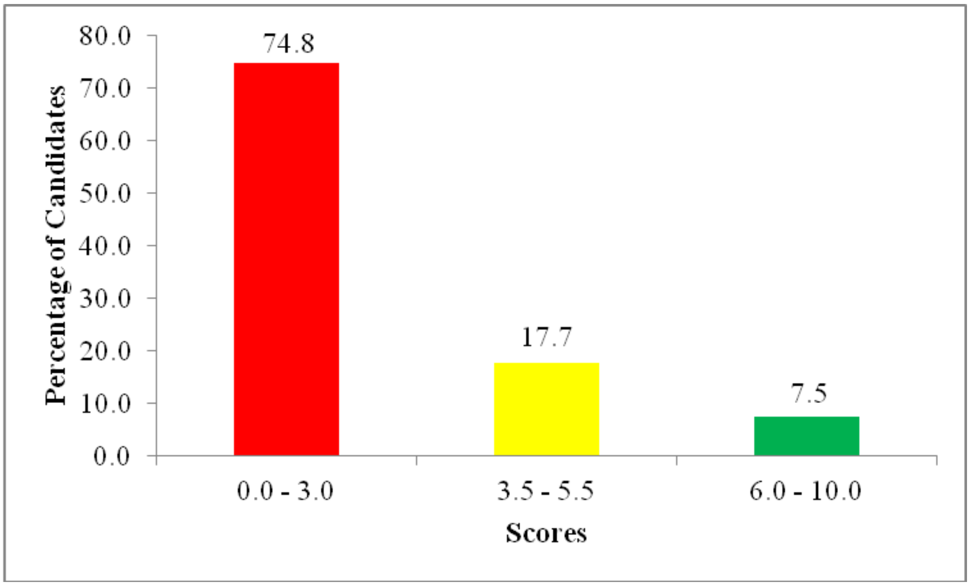


Figure 11: *The candidates' performance in question 10*

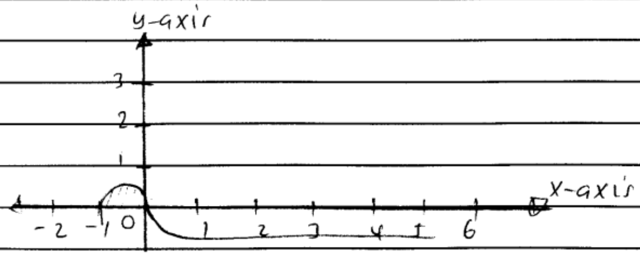
The analysis of the candidates' responses shows that the failure of candidates to get the correct answers was due to several reasons such as: In part (a), the candidates failed to utilize the condition $f(-1)f(0) < 0$ to prove that the value of x in the equation $x^3 - 6x^2 + 9x + 1 = 0$ is found between $x = -1$ and $x = 0$ that is found in the interval $x_1 < x < x_2$. Some candidates evaluated the expression

$\left[\frac{dy}{dx}\left(x^3-6x^2+9x+1\right)\right]_{-1}^{0}$ instead of evaluating $f(-1)$ and $f(0)$ while others computed the integral $\int_{-1}^{0}(x^3-6x^2+9x+1)dx$. Further analysis shows several candidates used the synthetic division method to find the remainder when x^3-6x^2+9x+1 is divided by x+1. Moreover, some candidates substituted the given values $x = -1$ and $x = 0$ into the derivatives $f^{'}(x)=3x^2-12x+9$ in testing for maximum and minimum points of the function x^3-6x^2+9x+1 instead of substituting the *x*-coordinates of the stationary points into the second derivatives $f^{''}(x)=6x-12$. There were also candidates who could not identify the x – intercept, y – intercept and stationary points. These were necessary steps in sketching the graph of $f\left(x\right)=x^3-6x^2+9x+1$. Other candidates in this category used inappropriate procedures such as using the table of values contrary to the requirements of the question. There were also candidates who sketched the graph of the first derivative $f^{'}(x)=3x^2-12x+9$ and second derivatives $f^{''}(x)=6x-12$ on the same *x, y* plane instead of sketching the graph of the function $f(x)=x^3-6x^2+9x+1$.

In part (b), these candidates failed to write correctly Taylor's series for $\left(x+h\right)^{\frac{1}{2}}$. Some candidates for example, quoted $(x+h)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2}xh + \frac{1}{4}h^2$ instead of $\left(x+h\right)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}h - \frac{1}{8}x^{-\frac{3}{2}}h^2 + \frac{1}{16}x^{-\frac{5}{2}}h^3$. Other candidates committed mistakes in writing the derivatives of $\left(x+h\right)^{\frac{1}{2}}$. These candidates gave answers such as: $f'(x)= \frac{1}{2}(x+h)^{-\frac{1}{2}}$, $f''(x)= -\frac{1}{4}(x+h)^{-\frac{3}{2}}$ and $f'''(x)= \frac{3}{8}(x+h)^{-\frac{5}{2}}$ instead of $f'\left(x\right)=\frac{1}{2}x^{-\frac{1}{2}}$, $f''\left(x\right)=\frac{-1}{4}x^{-\frac{3}{2}}$ and $f'''\left(x\right)=\frac{3}{8}x^{-\frac{5}{2}}$. There were some candidates who got part of the question correct because they presented the Taylor's series for $\left(x+h\right)^{\frac{1}{2}}$ but they failed to transform $\sqrt{10}$ into $\sqrt{9+1}$ which could be compared with $\left(x+h\right)^{\frac{1}{2}}$ to get $x = 9$ and $h = 1$. Extract 10.1 shows a sample solution from one of the candidates who performed poorly in this question.

10. (a) Solution.

$y = x^3 - 6x^2 + 9x + 1.$



$\int_a^b y\, dx.$

$\int_{-1}^{0} (x^3 - 6x^2 + 9x + 1)\, dx.$

$\int_{-1}^{0} \frac{x^4}{4} - \frac{6x^3}{3} + \frac{9x^2}{2} + 1x \; dx.$

$\int_{-1}^{0} \frac{x^4}{4} - \frac{6x^3}{3} + \frac{9x^2}{2} + x \; dx$

10. (a)

$\int_{-1}^{0} \frac{x^4}{4} - \frac{6x^3}{3} + \frac{9x^2}{2} + x.\, dx$

$\left.\frac{x^4}{4}\right|_{-1}^{0} - \left.\frac{6x^3}{3}\right|_{-1}^{0} + \left.\frac{9x^2}{2}\right|_{-1}^{0} + x.$

$(0 - 0.25) - (0 - 2) + (0 - 4.5) + x.$

$-0.25 + 2 - 4.5 + x.$

$= -2.75$

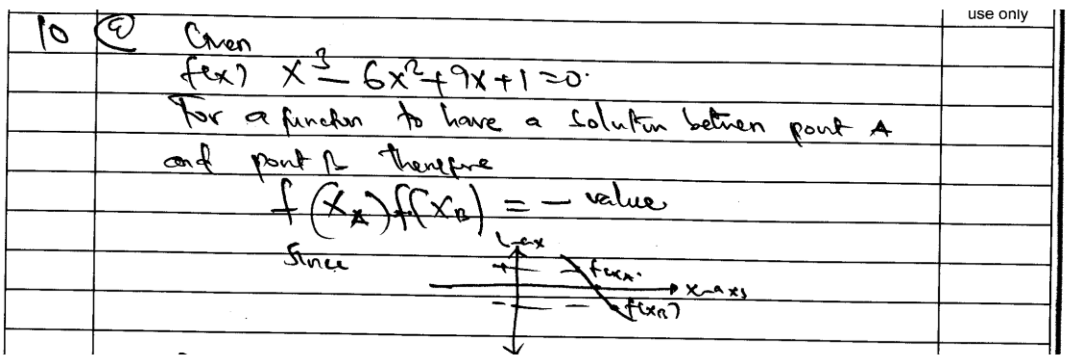
Extract 10.1: *An incorrect response to Question 10*

In Extract 10.2, the candidate formulated the integral using the given equation from $x = -1$ and $x = 0$. He/she also evaluated it to get -2.75 contrary to the requirements of the question.

Despite majority of candidates failing the question, there were 25 candidates (0.3%) who scored all 10 marks. In part (a), these candidates substituted $x = -1$ and $x = 0$ into the function $f(x) = x^3 - 6x^2 + 9x + 1$ to give $f(-1) = -15$ and

$f(0)=1$. Since $f(-1)f(0)<0$ they correctly confirmed that the equation has a solution between $x=-1$ and $x=0$. Thereafter, they equated the given equation to 0 to get the x – intercept at the point $(-0.1038, 0)$; entered $x=0$ into $f(x)=x^3-6x^2+9x+1$ to get y – intercept at the point $(0,1)$; they differentiated $f(x)$ to get $f'(x)=3x^2-12x+9$, they set the derivative equal to zero and solved the equation $3x^2-12x+9=0$ so as to obtain the x coordinates of the stationary points equal to 1 and 3, they substituted the values of *x* coordinates back into the original function to obtain the corresponding y – coordinates as 5 and 1; they differentiated the derivative of $f'(x)$ so as to arrive at $f''(x)=6x-12$. They also substituted $x=1$ and $x=3$ into the second derivative to give $f''(1)=-6$ and $f''(3)=6$. Since the second derivative $f''(1)=-6<0$ and $f''(3)=6>0$, the candidates correctly concluded that the local maximum and minimum occur at the points $(1,5)$ and $(3,1)$ respectively. Finally they used the local minimum, local maximum, x – intercept and y – intercept to sketch the graph of $y=x^3-6x^2+9x+1$ as illustrated in Extract 10.2.

The candidates who answered correctly part (b) substituted $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$ in Taylor's expansion of $(x+h)^{\frac{1}{2}}$ with $x^{\frac{1}{2}}$, $\frac{1}{2}x^{-\frac{1}{2}}$, $\frac{-1}{4}x^{-\frac{3}{2}}$ and $\frac{3}{8}x^{-\frac{5}{2}}$ getting $(x+h)^{\frac{1}{2}}=x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}h-\frac{1}{8}x^{-\frac{3}{2}}h^2+\frac{1}{16}x^{-\frac{5}{2}}h^3$. Then, they transformed $\sqrt{10}$ into $\sqrt{9+1}$ to get $x=9$ and $h=1$. Finally, they substituted $x=9$ and $h=1$ into the Taylor's expansion $x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}h-\frac{1}{8}x^{-\frac{3}{2}}h^2+\frac{1}{16}x^{-\frac{5}{2}}h^3$ to obtain the value of $\sqrt{10}$ is equal to 3.16229 correct to five decimal places.



$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 1$

$f(-1) = -15$

$f(0) = (0)^3 - 6(0) + 9(0) + 1$

$= 1$

~~$f(-1) \times f(0) = -15 \times 1 = -15$~~

Since $f(-1) \cdot f(0) =$ −value hence

The function has a solution between $x = -1$ and $x = 0$.

Given $y = x^3 - 6x^2 + 9x + 1$

For maximum and minimum values. $dy/dx = 0$

$dy/dx = 3x^2 - 12x + 9$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3$ and $x = 1$

at $x = 3, y = 1$ at $x = 1, y = 5$

for 2nd derivative

$d^2y/dx^2 = 6x - 12$ testing the roots

$(x-3)$ gives +ve $(x-1)$ give −ve

hence, $x = 3$ (minimum)

~~$(x=1)$ (maximum~~

~~09~~ 10 (a) The roots

if $f(x) = 0$

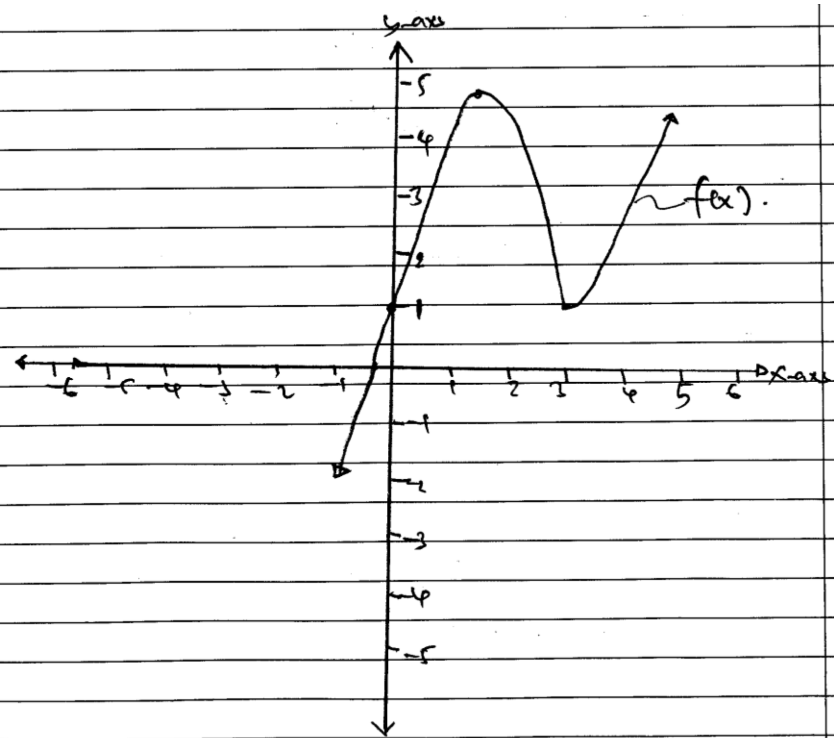
$x^3 - 6x^2 + 9x + 1 = 0$

On solving for roots

$x_1 = -0.1038$ (other roots are complex roots)

hence graph touches x-axis only once.

(b) Sketching the graph Also, for $x = 0$ $f(x) = 1$



INDEX NUMBER.................................. use only

10 (b) $(x+h)^{\frac{1}{2}}$

$a = x$.

Taylor theorem

$$f(a+h) = f(a) + \frac{h f'(a)}{1!} + \frac{h^2 f''(a)}{2!} + \frac{h^3 f'''(a)}{3!}$$

$f(x) = x^{\frac{1}{2}}$ $\quad f(x)$.

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

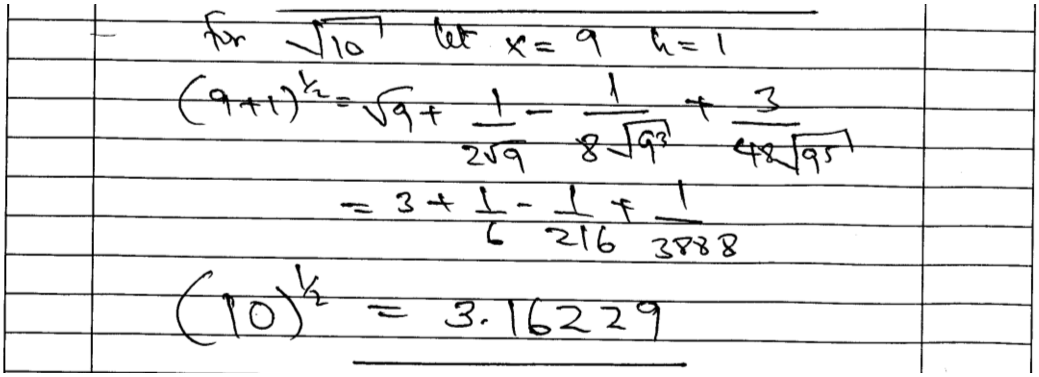
$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4\sqrt{x^3}}$

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} = \frac{+3}{8\sqrt{x^5}}$.

Hence

$$(x+h)^{\frac{1}{2}} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{4\sqrt{x^3}(2!)} + \frac{3h^3}{8\sqrt{x^5}(3!)}$$

$$(x+h)^{\frac{1}{2}} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8\sqrt{x^3}} + \frac{3h^3}{48\sqrt{x^5}}$$



for $\sqrt{10}$ let $x = 9$ $h = 1$

$(9+1)^{1/2} = \sqrt{9} + \frac{1}{2\sqrt{9}} - \frac{1}{8\sqrt{9^3}} + \frac{3}{48\sqrt{9^5}}$

$= 3 + \frac{1}{6} - \frac{1}{216} + \frac{1}{3888}$

$(10)^{1/2} = 3.16229$

Extract 10.2: *A sample of a correct response to Question 10*

Extract 10.2 shows that the candidate used the correct procedure to: show that the given equation has the root between $x = -1$ and $x = 0$, sketch the graph of the given polynomial function and use Taylor's theorem to expand the given expression.

2.2 142/2 ADVANCED MATHEMATICS 2

2.2.1 Question 1: Probability

The question was

(a) If A and B are such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$, calculate;

(i) $P(A \cap B')$.

(ii) $P(A / B')$

(b) Two dice are thrown simultaneously,

(i) list the sample space for the events.

(ii) find the probability that the sum of the numbers obtained on the dice is neither a multiple of 2 nor a multiple of 3.

(c) If X is binomially distributed, the probability that the event will happen exactly x times in n trials is given by function $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$.

Establish the validity of the Poisson approximation to the binomial distribution.

The analysis of data shows that 12,384 candidates (97.0%) attempted this question whereby 62.1 percent scored more than 5.0 marks and 0.4 percent scored all 15 marks. The question was therefore well performed. Figure 12 illustrates the summary of the candidates' performance in this question.

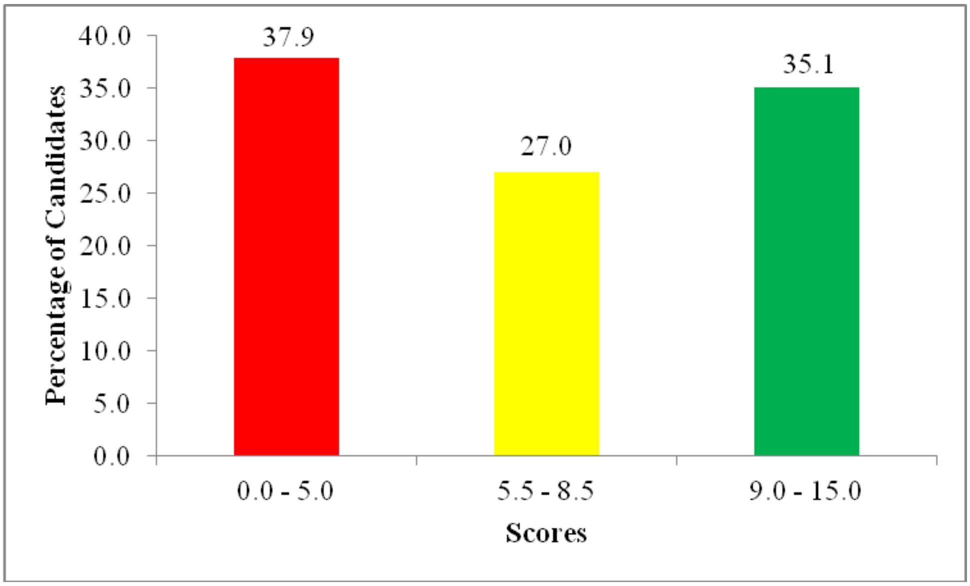


Figure 12 *Candidates' performance in question 1*

The analysis of candidates' responses shows that the majority of candidates who attempted this question had good performance. In part (a) (i), most candidates correctly substituted the probabilities $P(A \cup B)$ and $P(B)$ into the formula $P(A \cap B') = P(A \cup B) - P(B)$ to obtain $P(A \cap B')$ equal to $\frac{1}{4}$. In part (a) (ii), the candidates were able to find $P\left(A / B'\right)$ through the following steps: One, they substituted $P(B) = \frac{1}{4}$ into the formula $P(B') = 1 - P(B)$ to obtain $P\left(B\right) = \frac{3}{4}$; Two, they used the probabilities $P(A \cap B')$, $P(B')$ and the formula $P(A / B') = \frac{P(A \cap B')}{P(B')}$ to calculate the probability $P(A / B')$ giving the fraction $\frac{1}{3}$.

In part (b) (i), a number of candidates correctly used a table or tree diagram to list down the elements in the sample space of tossing two dices simultaneously. They correctly listed the required elements as {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5) and (6, 6). In part (b) (ii), most of these candidates were able to find the probability that the sum of numbers obtained on the dice is neither a multiple of 2 nor a multiple of 3 through the following steps: One, they computed the probability that the sum of two number is a multiple of 2 to get $P(A)=\frac{1}{2}$; Two, they computed the probability that the sum of two number is a multiple of 3 to obtain $P(B)=\frac{1}{3}$; Three, they calculated the probability that the sum of two numbers is a multiple of 2 and 3 as $P(A\cap B)=\frac{1}{6}$; Four, they correctly substituted the probabilities $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{3}$ and $P(A\cap B)=\frac{1}{6}$ obtained in step one, two and three into the formula $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ to calculate the probability that the sum of two number is a multiple of 2 or a multiple of 3 as $\frac{2}{3}$. Finally, they correctly used the formula $1-P(A\cup B)$ together with the probability $P(A\cup B)$ in step four to calculate the correct probability as $\frac{1}{3}$.

In part (c), the candidates demonstrated the following competences: Firstly, they recognised $\binom{n}{x}$ as $\frac{n(n-1)(n-2)\ldots\ldots(n-(x-1))}{x!}$ and $p=\frac{\lambda}{n}$; Secondly, they substituted the expression representing $\binom{n}{x}$ and p in step one into $\binom{n}{x}p^{x}(1-p)^{n-x}$ to get $\frac{n(n-1)(n-2)\ldots\ldots(n-(x-1))}{x!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$;

Thirdly, they carefully manoeuvred correctly the expression obtained step two to get $\dfrac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\left(\frac{x-1}{n}\right)\right)}{x!}\lambda^x\left(\left(1-\frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}\right)^{-\lambda}\left(1-\frac{\lambda}{n}\right)^{-x}$; Fourthly, they correctly used the condition $n \to \infty$ to simplify the expression $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\left(\frac{x-1}{n}\right)\right)$ to 1, $\left(1-\frac{\lambda}{n}\right)^{-x}$ to 1 and $\left(1-\frac{\lambda}{n}\right)^{\frac{-n}{\lambda}}$ to e. Finally, they substituted correctly the results obtained in step four to establish the validity of the Poisson approximation to the binomial distribution as $P(x,\lambda)=\dfrac{\lambda^x e^{-\lambda}}{x!}$. Extract 11.1 shows the solution from one of the candidates who answered correctly this question.

1 (a) Soln

$P(A) = \frac{1}{3}$ $\quad P(B) = \frac{1}{4}$ $\quad P(A \cup B) = \frac{1}{2}$

$P(A \cap B')$

from,

$P(A) = P(A \cap B) + P(A \cap B')$

$P(A \cap B') = P(A) - P(A \cap B)$ —— (i)

from $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{1}{2} = \frac{1}{3} + \frac{1}{4} - P(A \cap B)$

$\therefore P(A \cap B) = \frac{1}{12}$

1 (a) Substituting in eqn (i)

$P(A \cap B') = \frac{1}{3} - \frac{1}{12}$

$= \frac{1}{4}$

Hence,

$P(A \cap B') = \frac{1}{4}$

1 (a) (i) Soln

$P(A/B')$

$$P(A/B') = \frac{P(A \cap B')}{P(B')}$$

$P(A \cap B') = 1/4$

$P(B') = 1 - P(B)$

$P(B') = 1 - 1/4 = 3/4$

$$P(A/B') = \frac{1/4}{3/4} = 1/3$$

Hence

$P(A/B') = 1/3$

1 (b) Soln

(i) let sample space be S

from,

Dice 1/2	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

Thus

S = {(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (1,2),
(2,2), (3,2), (4,2), (5,2), (6,2), (1,3), (2,3),
(3,3), (4,3), (5,3), (6,3), (1,4), (2,4), (3,4),
(4,4), (5,4), (6,4), (1,5), (2,5), (3,5), (4,5),
(5,5), (6,5), (1,6), (2,6), (3,6), (4,6), (5,6),
(6,6)}

(ii) Soln

Let multiple of 2 be A

Multiple of 3 be B

~~P(Neither A nor B) = P(A∪B)'~~

Solving for (A∪B)

$A = \{(1,1), (3,1), (5,1), (2,2), (4,2), (6,2), (1,3), (3,3), (5,3), (2,4), (4,4), (6,4), (1,5), (3,5), (5,5), (2,6), (4,6), (6,6)\}$

$B = \{(2,1), (5,1), (1,2), (4,2), (3,3), (6,3), (2,4), (5,4), (1,5), (4,5), (3,6), (6,6)\}$

1(b)
(a)

A∪B = {(1,1), (3,1), (5,1), (2,2), (4,2), (6,2), (1,3), (3,3), (5,3), (2,4), (4,4), (6,4), (1,5), (3,5), (5,5), (2,6), (4,6), (6,6), (2,1), ~~(5,1)~~, (1,2), ~~(4,2)~~, ~~(3,3)~~, (6,3), ~~(2,4)~~, (5,4), ~~(1,5)~~, (4,5), (3,6), ~~(6,6)~~}

$n(A \cup B) = 24$

$n(S) = 36$

$P(A \cup B) = \frac{n(A \cup B)}{n(S)}$

$= \frac{24}{36}$

~~P(A∪B)~~ $= \frac{2}{3}$

$P(A \cup B)' = 1 - P(A \cup B) \quad \left[1 - \frac{2}{3}\right]$

$= 1 - \frac{2}{3} \quad = \frac{1}{3}$

$= \frac{1}{3}$

Hence Probability $= \frac{1}{3}$

1. (c) Soln

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(x) = {}^nC_x p^x (1-p)^{n-x}$$

$$P(x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

$$P(x) = \frac{n(n-1)(n-2)(n-3)(n-4)\cdots(n-x+1)(n-x)!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

($\lambda \equiv$ Mean)

also, $\lambda = np$ $\quad p = \lambda/n$

$$P(x) = \frac{n(n-1)(n-2)(n-3)\cdots(n-x+1)\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}}{x!}$$

$$P(x) = \frac{n\,n(1-1/n)\,n(1-2/n)\,n(1-3/n)\cdots n\left(1+\frac{1-x}{n}\right)\lambda^x\left(1-\frac{\lambda}{n}\right)^n}{x!\,n^x(1-\lambda/n)^x}$$

$$P(x) = \frac{n^x(1-1/n)(1-2/n)(1-3/n)\cdots\left(1+\frac{1-x}{n}\right)\lambda^x(1-\lambda/n)^n}{x!\,n^x(1-\lambda/n)^x}$$

as $n \to \infty$

$$P(x) = \frac{(1-1/\infty)(1-2/\infty)(1-3/\infty)\cdots\left(1+\frac{1-x}{\infty}\right)\lambda^x(1-\lambda/n)^n}{x!\,(1-\lambda/\infty)^x}$$

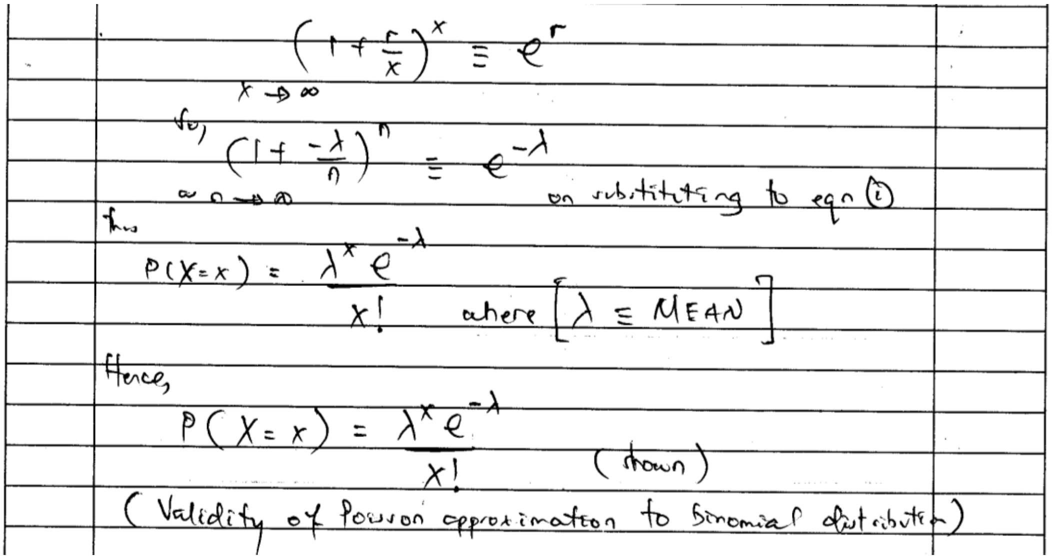
thus $\left(1/\infty \equiv 0\right)$

1(c) $$P(x) = \frac{1(1)(1)(1)\cdots(1)\,\lambda^x\left(1+\frac{-\lambda}{n}\right)^n}{(1)x!} \quad \text{— (i)}$$

~~but~~ from,

$$\left(1+\frac{r}{x}\right)^x$$

as $x \to \infty$



$$\left(1+\frac{r}{x}\right)^x \equiv e^r$$
$x \to \infty$

so,
$$\left(1+\frac{-\lambda}{n}\right)^n \equiv e^{-\lambda}$$
$n \to \infty$ on substituting to eqn (i)

thus
$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{where } [\lambda \equiv \text{MEAN}]$$

Hence,
$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{(shown)}$$

(Validity of Poisson approximation to binomial distribution)

Extract 11.1: *A correct response to question 1*

In Extract 11.1, the candidate was able to use the properties of probability theorems to calculate $P(A \cap B')$ and $P(A/B')$. and used matrix table to get the required solution. Furthermore, the candidate was able to use binomial distribution formula to approximate Poisson formula.

Despite the good performance in this question there were candidates (37.9%) who scored low marks. These candidates faced some challenges in answering the question as follows: In part (a), some candidates wrote incorrect formulae, such as $P\left(\frac{A}{B'}\right) = \frac{P(A \cap B')}{P(B)}$, $P(A \cap B') = P(B) - P(A \cap B)$ and $P(A \cap B') = P(A).P(B')$.

Hence, they ended up getting incorrect answers such as $\frac{2}{3}$, $\frac{1}{6}$ and $\frac{1}{4}$. In part (b), they failed to differentiate between a dice and a coin. There were candidates for example who used a tree diagram to list the elements in the sample space as {HH, HT, TH and TT}. Other candidates quoted the elements in the first and second die as {1, 2, 3, 4, 5, 6} and {1, 2, 3, 4, 5, 6} which was contrary to the demand of the question. Hence, they could not provide the correct answers in (ii). In part (c), the candidates failed to apply the concept of binomial distribution to establish the Poisson formula. Some of them for example just wrote if $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

then $\lambda \approx \mu$. Other candidates mentioned the conditions for Poisson distribution instead of proving as it was required in the question. A sample solution of an incorrect answer from a candidate who performed poorly in part (c) of this question is shown in Extract 11.2.

1. (c) Consider when the number of trials is very greater example; $n = 100,000$.
Also probability of success is very small such that $p = 0.0001$.
Now from binomial distribution formular,

say $X = 1$

$P(X = x)$

$$P(X = 1) = {}^{100,000}C_1 \left[(0.0001)^1 (1 - 0.0001)^{99999}\right]$$

$$P(X = 1) = 0.000454.$$

By Poisson.
From poison formular

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\lambda = np$$

$$\lambda = 100000 \times 0.0001$$

$$\lambda = 10.$$

$$P(X = 1) = \frac{e^{-10} \times (10)^1}{1!}$$

$$P(X = 1) = 0.000454$$

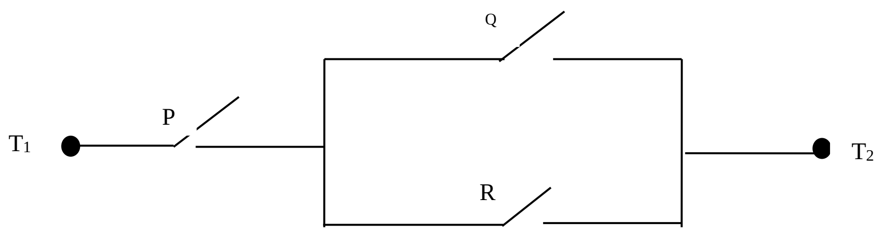
Since when use binomial distribution gives the same results as when using poisson ~~approxi~~ formula hence the binomial distribution can be approximated to poisson under the given conditions above,

Extract 11.2: *An incorrect response to Question 1 (c)*

In Extract 11.2, the candidate solved $P(X = 1)$ using both the binomial distribution and Poisson approximation formula where $P = 0.0001$ and $n = 100,000$. This was incorrect because the question required him/her to establish the validity of the Poisson approximation to the binomial distribution.

2.2.2 Question 2: Logic

The question had three parts (a), (b) and (c). In part (a), the candidates were given that the contrapositive of Y is given by $\sim(Q \wedge P) \rightarrow \sim P$. By using the laws of algebra of propositions, they were required to show that its inverse is a tautology. In part (b), they were required to test the validity of the argument whose conclusion is $\sim Q$ and premises are $P \rightarrow (\sim P \rightarrow Q), Q \rightarrow \sim P$, and P. In part (c) (i), the candidates were required to construct a truth table for the compound statement that corresponds to the following circuit:



In part (c) (ii), they were required to draw a simple network diagram for the statement $(P \rightarrow Q) \wedge (P \vee Q)$.

This question was opted by 12,751 (99.8%) candidates. The analysis shows that 10,121 (79%) candidates scored more than 5.0 marks and some candidates 792 (6.2%) scored all 15 marks. Based on this analysis, the candidates' performance in Question 13 was good. Figure 13 summarizes the candidates' performance in this question.

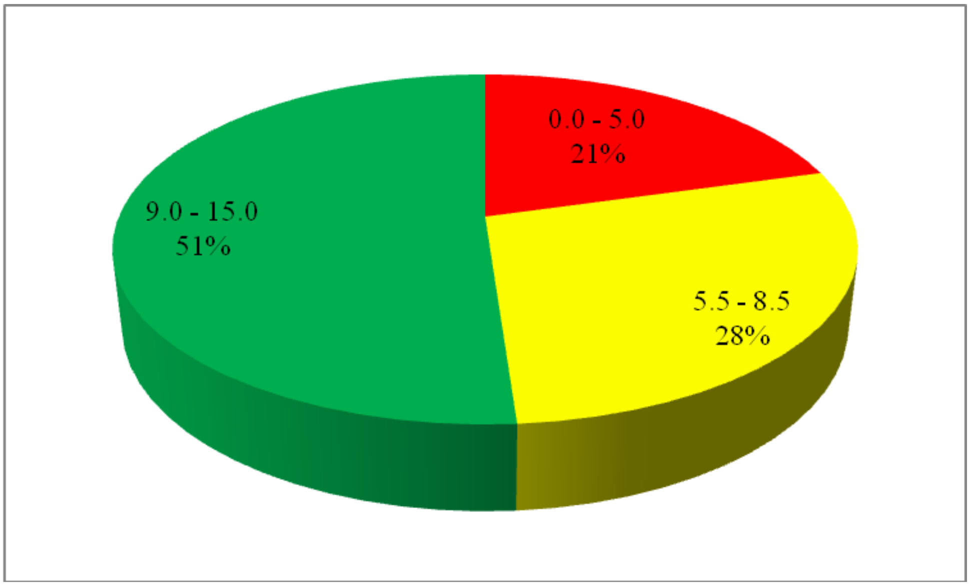


Figure 13: *Candidates' performance in question 2*

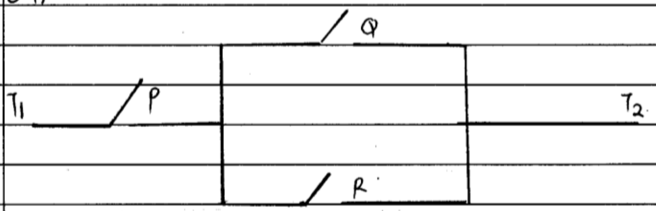
The analysis of candidates' responses shows that majority of the candidates (51%) had good performance. Majority of candidates demonstrated the following strengths: In part (a), the candidates managed to write the inverse of statement Y as $\sim P \rightarrow \sim(Q \wedge P)$. They then used the laws of algebra of propositions to show that the inverse of y is tautology. In part (b), the candidates were able to formulate the compound statement $\left[\left(P \rightarrow (\sim P \rightarrow Q)\right) \wedge (Q \rightarrow \sim P) \wedge P\right] \rightarrow \sim Q$ from the given premises and conclusion. From this compound statement, the candidates used either a truth table or the laws of algebra of propositions to test its validity. In both cases, they ended up with the truth value T which confirms that the argument is valid. In part (c) (i), the candidates were able to formulate the compound statement from the given electrical network that is $P \wedge (Q \vee R)$. They also correctly constructed its truth table. In part (c) (ii), they simplified the statement $(P \rightarrow Q) \wedge (P \vee Q)$ by using laws of algebra of proposition and got Q as the conclusion and finally drew its network diagram. Extract 12.1 illustrates a correct solution from one of the candidates.

2 b) $[(P \rightarrow (\sim P \rightarrow Q)) \wedge [(Q \rightarrow \sim P) \wedge P]] \rightarrow \sim Q$

P	Q	$\sim P$	$\sim Q$	a: $\sim P \rightarrow Q$	b: $P \rightarrow a$	c: $Q \rightarrow \sim P$	d: $c \wedge P$	e: $b \wedge d$	$e \rightarrow \sim Q$
T	T	F	F	T	T	F	F	F	T
T	F	F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F	F	T
F	F	T	T	F	T	T	F	F	T

Since the last column of the truth table is a tautology, hence the arguement is valid.

c i)



$\equiv P \wedge (Q \vee R)$

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

2 c ii) $(P \rightarrow Q) \wedge (P \vee Q)$ ... Given

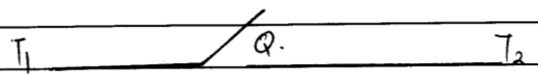
$\equiv (\sim P \vee Q) \wedge (P \vee Q)$ ... Implicational law

$\equiv (Q \vee \sim P) \wedge (Q \vee P)$ ... Commutative law

$\equiv Q \vee (\sim P \wedge P)$ ... Distributive law

$\equiv Q \vee F$ ... Complementary law

$\equiv Q$ ... identity law



Extract 12.1: *A correct response to question 2*

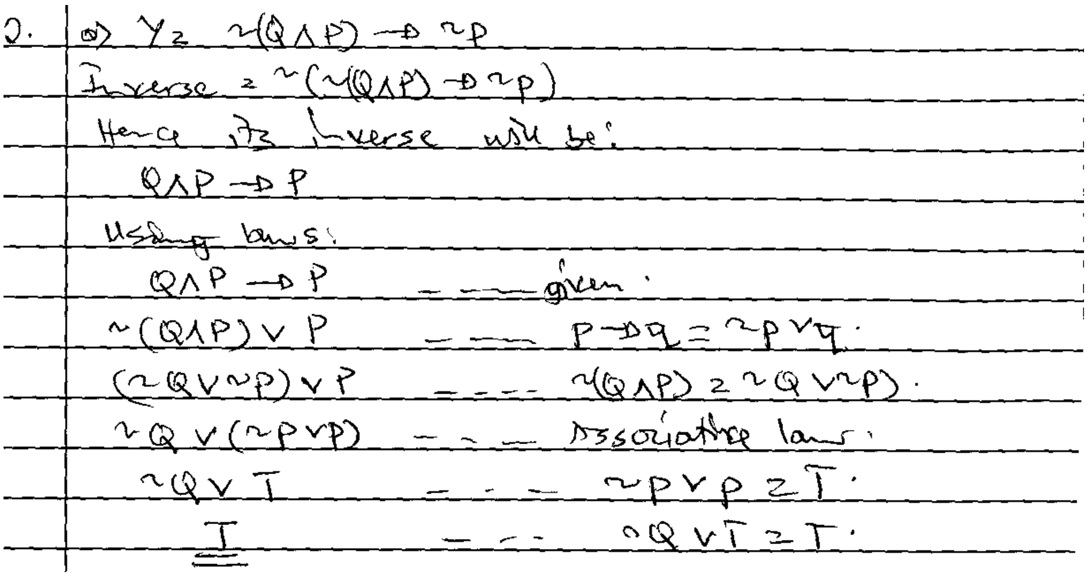
In Extract 12.1, the candidate demonstrated the ability to: simplify the given statements by using the laws of algebra of propositions; test the validity of given argument; construct the truth table from the given electric network and to draw a simple network diagram from the statement.

Although majority of candidates performed well, there were some candidates (21%) who scored low marks in this question. The following are some of the challenges they encountered: In part (a), they tested the validity of the inverse of the contrapositive instead of the inverse of statement Y which is $\sim P \to \sim (Q \wedge P)$. as the result they ended up testing for the validity of $(Q \wedge P) \to P$ which was wrong. Some of these candidates tested the validity of the converse $(Q \wedge P) \to P$ as they could not differentiate the term converse from inverse. In simplifying the statement $\sim P \to \sim (Q \wedge P)$, many of these failed to distinguish between commutative and associative laws. There were also a good number of candidates who stated incorrectly some basic laws of algebra of proposition. For example, they wrote $\sim P \to \sim (Q \wedge P) \equiv P \vee (\sim Q \vee \sim P)$ by basic property. Some of these candidates applied the laws of set instead of logic laws, for example the complement instead of negation.

In part (b), there were some candidates who failed to connect the given premises by using the conjunction $(\wedge)$ and implication $(\to)$ to get $\left[(P \to (\sim P \to Q)) \wedge (Q \to \sim P) \wedge P\right] \to \sim Q$. These candidates were unable to test the validity of the argument. Other candidates drew the truth table of each given premise but they did not conclude whether the argument is valid or not.

The candidates who answered part (c) (i) incorrectly formulated incorrect compound statements from the given electric network like $P \vee (P \wedge Q)$; and in part (ii), they transformed the given proposition to $(\sim P \vee Q) \wedge (P \vee Q)$ using implication law and finally constructed it's electric circuit. Other candidates in this category defined the implication connective incorrectly and ended up with

$(\sim P \vee Q) \wedge (P \wedge Q)$ which led to simple statement *P* instead of *Q*. Hence, the candidates drew a circuit for *p* which was incorrect. Extract 12.2 is an example of an incorrect solution from one of the candidates.


2. (a) Y = ~(Q∧P) → ~P
Inverse = ~(~(Q∧P) → ~P)
Hence its inverse will be:
Q∧P → P
Using laws:
Q∧P → P ——— given.
~(Q∧P) ∨ P ——— P→Q = ~P∨Q.
(~Q∨~P)∨P ——— ~(Q∧P) = ~Q∨~P).
~Q∨(~P∨P) ——— Associative laws.
~Q∨T ——— ~P∨P = T.
T ——— ~Q∨T = T.

Extract 12.2: *A sample of an incorrect answer to Question 2 (a)*

In Extract 12.1, the candidate found the inverse of the contrapositive statement instead of the inverse of statement *Y* to show that the inverse is a tautology by using the laws of algebra of propositions.

2.2.3 Question 3: Vectors

This question consisted of parts (a), (b) and (c). In part (a), the candidates were required to find a unit vector orthogonal to both vectors $\underline{a}$ and $\underline{b}$, given that $\underline{a} = \underline{i} - \underline{j} + 2\underline{k}$ and $\underline{b} = \underline{i} + \underline{j}$. In part (b), the candidates were given that the position vectors of the points A and B are $2\underline{i} + 3\underline{j} - \underline{k}$ and $-\underline{i} + 5\underline{j} + 5\underline{k}$ respectively. They were required to find the position vector of point C. In part (c), they were required to use the cosine rule to show that in the triangle ABC, $\underline{c} = \underline{b}\cos A + \underline{a}\cos B$.

A total of 12,609 candidates (98.7%) responded to the question whereby 5,458 candidates (43%) scored more than 5.0 marks. This shows that the candidates' performance in this question was average as shown in Figure 14.

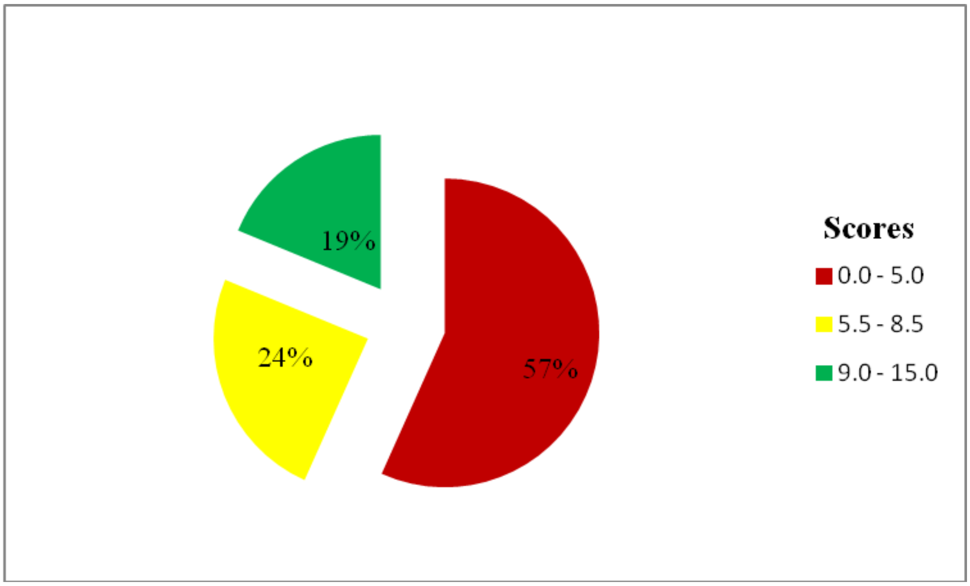


Figure 14: *Candidates' performance in question 3*

The analysis of the candidates' response shows that, the candidates (19%) who scored high marks had adequate knowledge of Vectors. In part (a), they were able to use the formula $\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$ to get a unit vector orthogonal to $\underline{a}$ and $\underline{b}$ as $\frac{\sqrt{3}}{3}\left(-\underline{i} + \underline{j} + \underline{k}\right)$. In part (b), they correctly substituted $\underline{a} = 2\underline{i} + 3\underline{j} - \underline{k}$, $\underline{b} = -\underline{i} + 5\underline{j} + 7\underline{k}$, $m = 2$ and $n = 1$ into the formula $\underline{c} = \frac{m\underline{b} + n\underline{a}}{m + n}$ to obtain the position vector of point C is equal to $\underline{c} = \frac{13}{3}\underline{j} + \frac{13}{3}\underline{k}$. In part (c), they were able to show that $\underline{c} = \underline{b}\cos A + \underline{a}\cos B$ through the following steps: Firstly, they correctly applied the cosine rule on triangle ABC to formulate the equation relating $\underline{a}$, $\underline{b}$, $\underline{c}$, A, B, and C as $|\underline{a}|^2 = |\underline{b}|^2 + |\underline{c}|^2 - 2|\underline{b}||\underline{c}|\cos A$ and $|\underline{b}|^2 = |\underline{a}|^2 + |\underline{c}|^2 - 2|\underline{a}||\underline{c}|\cos B$; Secondly, they added the equations in step one to obtain $\underline{c} = \underline{b}\cos A + \underline{a}\cos B$. Extract 13.1 is a sample solution from of one the candidates who answered correctly this question.

3 a) given

$\underline{a} = \underline{i} - \underline{j} + 2\underline{k}$

$\underline{b} = \underline{i} + \underline{j} + 0\underline{k}$

the unit vector orthogonal to both $\underline{a}$ and $\underline{b}$

let ~~tha~~ - the vector be $\underline{c}$

$\underline{c} = \underline{a} \times \underline{b}$

$$\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$\underline{c} = \underline{i}(0-2) - \underline{j}(0-2) + \underline{k}(1+1)$

$\underline{c} = -2\underline{i} + 2\underline{j} + 2\underline{k}$

the unit vector $\underline{c}$

$$= \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{\sqrt{4+4+4}}$$

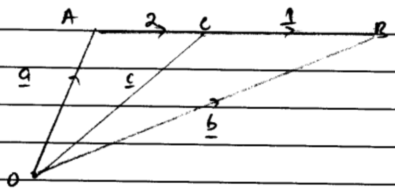
$$= \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{\sqrt{12}} = \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{2\sqrt{3}}$$

use only

3 a) $\hat{c} = \frac{-1}{\sqrt{3}}\underline{i} + \frac{1}{\sqrt{3}}\underline{j} + \frac{\underline{k}}{\sqrt{3}}$

∴ The unit vector orthogonal to given $\underline{a}$ and $\underline{b}$ is $\frac{-1}{\sqrt{3}}\underline{i} + \frac{1}{\sqrt{3}}\underline{j} + \frac{1}{\sqrt{3}}\underline{k}$

3 (b)



$$\frac{\overline{AC}}{\overline{CB}} = \frac{2}{1}$$

$$\frac{\overrightarrow{OC} - \overrightarrow{OA}}{\overrightarrow{OB} - \overrightarrow{OC}} = \frac{2}{1}$$

$2\overrightarrow{OB} - 2\overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$2\overrightarrow{OB} + \overrightarrow{OA} = 3\overrightarrow{OC}$$

$$\frac{2\overrightarrow{OB} + \overrightarrow{OA}}{3} = \overrightarrow{OC}$$

$$= \frac{2(-\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}) + 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}}{3}$$

$$= \frac{-2\underset{\sim}{i} + 10\underset{\sim}{j} + 14\underset{\sim}{k} + 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}}{3}$$

$$\overrightarrow{OC} = \frac{13\underset{\sim}{j}}{3} + \frac{13\underset{\sim}{k}}{3} = \text{position vector of point C}$$

3 (c) cosine rule

$$|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{b}\cos C$$

$$|\underset{\sim}{c}|^2 = |\underset{\sim}{c}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{c}\cdot\underset{\sim}{b}\cos A + |\underset{\sim}{c}|^2 + |\underset{\sim}{a}|^2 - 2\underset{\sim}{c}\cdot\underset{\sim}{a}\cos B - 2\underset{\sim}{a}\cdot\underset{\sim}{b}\cos C$$

$$|\underset{\sim}{c}|^2 + 2\underset{\sim}{a}\underset{\sim}{b}\cos C = 2|\underset{\sim}{c}|^2 + |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2(\underset{\sim}{c}\underset{\sim}{b}\cos A + \underset{\sim}{c}\underset{\sim}{a}\cos B)$$

$$|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 = 2|\underset{\sim}{c}|^2 + |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2(\underset{\sim}{c}\cdot\underset{\sim}{b}\cos A + \underset{\sim}{c}\underset{\sim}{a}\cos B)$$

$$2(\underset{\sim}{c}\cdot\underset{\sim}{b}\cos A + \underset{\sim}{c}\cdot\underset{\sim}{a}\cos B) = 2|\underset{\sim}{c}|^2$$

but $|\underset{\sim}{c}|^2 = \underset{\sim}{c}\cdot\underset{\sim}{c}$

$$\underset{\sim}{c}\cdot\underset{\sim}{b}\cos A + \underset{\sim}{c}\cdot\underset{\sim}{a}\cos B = \underset{\sim}{c}\cdot\underset{\sim}{c}$$

dividing by $\underset{\sim}{c}$ throughout we get

$$\underset{\sim}{b}\cos A + \underset{\sim}{a}\cos B = \underset{\sim}{c}$$

∴ Hence proved,

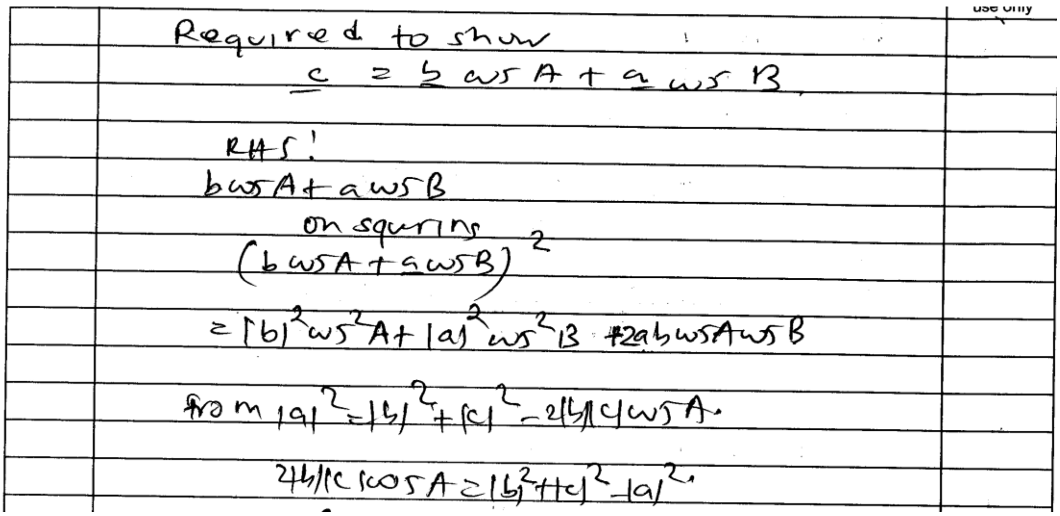
Extract 13.1: *A sample of a correct answer to Question 3*

In Extract 13.1, the candidate demonstrated competence in responding to the question on Vectors.

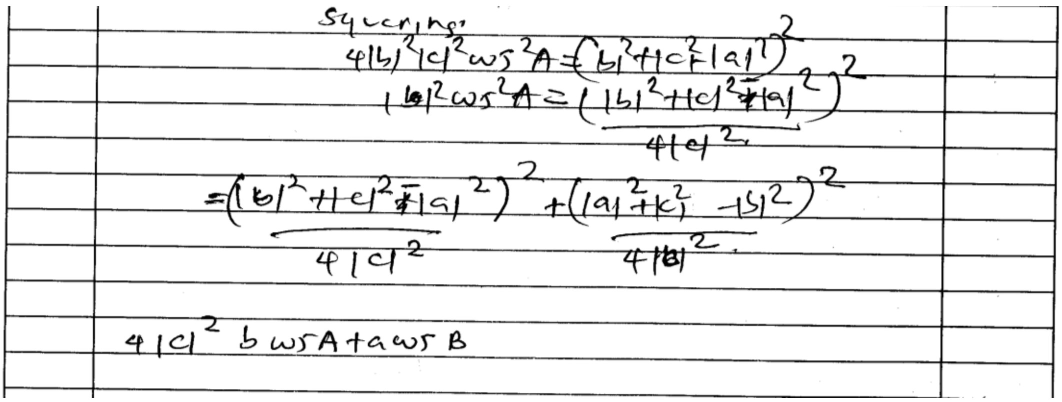
Further analysis of data shows that 57 percent of candidates scored low marks. The candidates in this category faced some difficulties in attempting this question as follows: In part (a), some of these candidates were able to write the cross product of vectors $\underline{a}$ and $\underline{b}$ but committed mistakes in doing basic operations such as addition and subtraction. One candidate for example, multiplied the given vectors to get $-2\underline{i} - 2\underline{j} + 2\underline{k}$ instead of $-2\underline{i} + 2\underline{j} + 2\underline{k}$, whereby only the coefficient of $\underline{j}$ was incorrect. Others computed the dot product of vector $\underline{a}$ and $\underline{b}$ to get 0.

Thereafter, they concluded that the given vectors are orthogonal contrary to the requirements of the question. Additionally, there were candidates who used incorrect formulae like $\frac{\underline{a}+\underline{b}}{|\underline{a}+\underline{b}|}$. Such formulae gave problems in the later part as most candidates ended up getting incorrect unit vector such as $\frac{\underline{i}+\underline{j}}{\sqrt{2}}$.

In part (b), most of these candidates were unable to maintain the direction of the resulting vector with an arrow ® when C divides $\overrightarrow{AB}$ internally. Hence, they related these vectors to get $\frac{\overrightarrow{AC}}{\overrightarrow{BC}}=\frac{\underline{c}-\underline{a}}{\underline{c}-\underline{b}}=\frac{2}{1}$ instead of $\frac{\overrightarrow{AC}}{\overrightarrow{CB}}=\frac{\underline{c}-\underline{a}}{\underline{b}-\underline{c}}=\frac{2}{1}$. Thereafter, they substituted $\underline{a}=2\underline{i}+3\underline{j}-\underline{k}$ and $\underline{b}=-\underline{i}+5\underline{j}+7\underline{k}$ into the relation $\frac{\underline{c}-\underline{a}}{\underline{c}-\underline{b}}=\frac{2}{1}$ to obtain $\underline{c}=-4\underline{i}+7\underline{j}+15\underline{k}$ which is incorrect. In part (c), most of these candidates formulated incorrect formula to relate sides and angles of triangle ABC. Hence, they failed to show that $\underline{c}=\underline{b}\cos A+\underline{a}\cos B$. The most common wrong formulae given included $|\underline{a}|^2=|\underline{b}|^2+|\underline{c}|+2|\underline{b}||\underline{c}|\cos A$ and $|\underline{c}|^2=|\underline{a}|^2+|\underline{b}|^2+2|\underline{b}||\underline{a}|\cos C$. Extract 13.2 is a sample of an incorrect answer which had some mistakes committed by the candidates in answering part (c) of this question.


Required to show
c = b cos A + a cos B

RHS:
b cos A + a cos B
on squaring
(b cos A + a cos B)²
= |b|² cos² A + |a|² cos² B +2ab cos A cos B
from |a|² = |b|² + |c|² − 2|b||c| cos A.
2|b||c| cos A = |b|² + |c|² − |a|².


Squaring:

$4|b|^2|c|^2\cos^2 A = (|b|^2+|c|^2-|a|^2)^2$

$|b|^2\cos^2 A = \frac{(|b|^2+|c|^2-|a|^2)^2}{4|c|^2}$

$= \frac{(|b|^2+|c|^2-|a|^2)^2}{4|c|^2} + \frac{(|a|^2+|c|^2-|b|^2)^2}{4|b|^2}$

$4|c|^2 \; b\cos A + a\cos B$

Extract 13.2 *A sample of an incorrect response to Question 3 (c)*

In Extract 13.2, the candidate was unable to use the cosine rule to show that $\underset{\sim}{c} = \underset{\sim}{b}\cos A + \underset{\sim}{a}\cos B$.

2.2.4 Question 4: Complex Numbers

This question had parts (a), (b) and (c). In part (a), the candidates were given that $2x + 10yi - 4y = -12 + 5i$ and required to find the values of x and y. In part (b), they were required to express $(\cos\theta + i\sin\theta)^{-n}$ in the form $a + ib$. In part (c), they were given that $z = x + iy$ and required to express the complex number $\frac{z+i}{iz+2}$ in polynomial form and hence find the resulting complex number when $z = 1 + 2i$.

The analysis of data indicates that 12,539 candidates (98.2%) attempted the question, whereby 10,337 candidates (82%) scored from 5.5 to 15.0 marks. Therefore, the general performance of the candidates in this question was good. Figure 15 shows the percentage distribution of candidates who got weak, average and good scores.

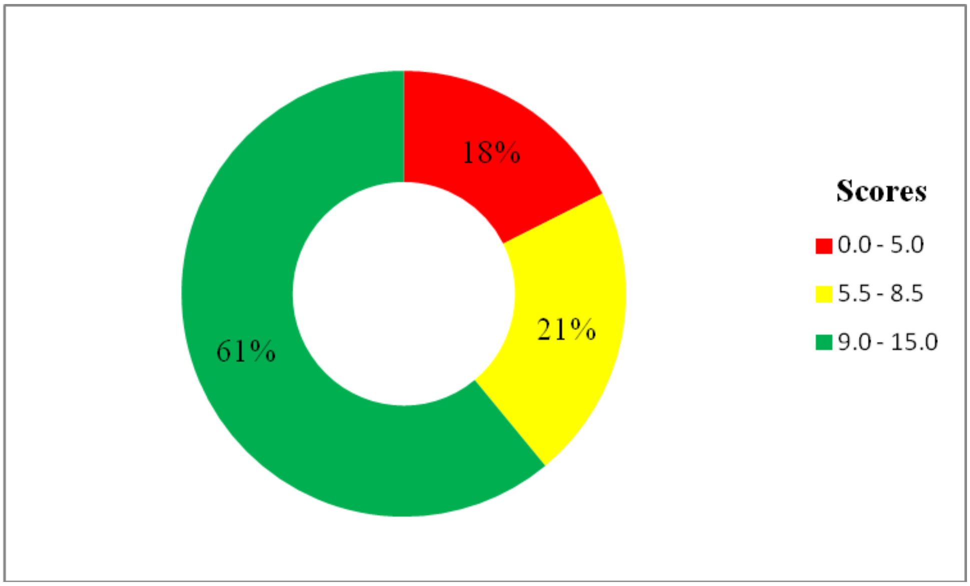


Figure 15: *Candidates' performance in question 4*

In Figure 15, it is shown that 61 percent of the candidates scored high marks. In part (a), these candidates showed a good understanding of the concepts tested as they correctly compared the real and imaginary parts of the equation $2x+10yi-4y=-12+5i$ to get the equations $2x-4y=-12$ and $10y=5$. In addition, they computed the equations in step one to get $x=-5$ and $y=\frac{1}{2}$. In part (b), the candidates correctly applied De Moivres theorem to express $(\cos\theta+i\sin\theta)^{-n}$ in the form $a+ib$ as $\cos n\theta-\sin n\theta$. In part (c), the candidates correctly substituted $z=x+iy$ into the expression $\frac{z+i}{iz+2}$ to get the expression $\frac{x+(y+1)i}{2-y+ix}$. Thereafter, they correctly rationalized the expression in step 1 to obtain $\frac{3x}{(2-y)^2+x^2}+\frac{i(y-y^2-x^2+2)}{(2-y)^2+x^2}$. From $z=1+2i$, they were able to identify the values of x and y as 1 and 2 respectively. They then correctly substituted the values of x and y into $\frac{3x}{(2-y)^2+x^2}+\frac{i(y-y^2-x^2+2)}{(2-y)^2+x^2}$ to get the complex number $3-i$ as required. Extract 14.1 shows a sample of responses from one of the candidates who correctly answered this question.

4 a) $2x + 10yi - 4y = -12 + 5i$

$(2x - 4y) + 10yi = -12 + 5i$

$2x - 4y = -12 \quad \ldots \text{i)}$

$10y = 5 \quad \ldots \text{ii)}$

$y = \frac{5}{10}$

$y = \frac{1}{2}$.

$2x - 4y = -12$

$2x - 4\left(\frac{1}{2}\right) = -12$

$2x - 2 = -12$

$2x = -12 + 2$

$2x = -10$

$x = -5$

$\therefore$ The value of x is -5 and y is $\frac{1}{2}$.

b) $(\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n)\theta + i\sin(-n)\theta$

$= \cos n\theta - i\sin n\theta$

c) $z = x + iy$

$\frac{z+i}{iz+2} = \frac{x+iy+i}{i(x+iy)+2}$

$= \frac{x+i(y+1)}{ix+i^2y+2}$

$= \frac{x+i(y+1)}{ix-y+2}$

$= \frac{x+i(y+1)}{(2-y)+ix}$

4 c) $= \frac{x+i(y+1)}{(2-y)+ix} \times \frac{(2-y)-ix}{(2-y)-ix}$

$= \frac{x(2-y) - ix^2 + i(2-y)(y+1) - i^2x(y+1)}{(2-y)^2 - (ix)^2}$

$= \frac{x(2-y) - ix^2 + i(2-y)(y+1) + x(y+1)}{(2-y)^2 + x^2}$

$$= \frac{(x(2-y)+x(y+1)) + i((2-y)(y+1) - x^2)}{(2-y)^2 + x^2}$$

$$= \frac{(2x - xy + xy + x) + i(2y + 2 - y^2 - y - x^2)}{(2-y)^2 + x^2}$$

$$= \frac{3x + i(y - y^2 - x^2 + 2)}{(2-y)^2 + x^2}$$

$$= \frac{3x}{(2-y)^2 + x^2} + \frac{i(y - y^2 - x^2 + 2)}{(2-y)^2 + x^2}$$

$$\therefore \frac{z+i}{iz+2} = \frac{3x}{(2-y)^2 + x^2} + \frac{i(y - y^2 - x^2 + 2)}{(2-y)^2 + x^2}$$

When $z = 1 + 2i$

$x = 1,\ y = 2.$

$$\frac{z+i}{iz+2} = \frac{3(1)}{(2-2)^2 + 1^2} + \frac{i(2 - 2^2 - 1^2 + 2)}{(2-2)^2 + 1^2}$$

$$= \frac{3}{1} + \frac{i(-1)}{1}$$

$$= 3 - i$$

$$\therefore \frac{z+i}{iz+2} = 3 - i$$

Extract 14.1: *A sample of a correct answer to Question 4*

In Extract 14.1, the candidate was able to use basic operations of complex numbers in solving the values of x and y in part (a). In part (b), the candidate correctly used the De Moivre's theorem. In part (c), the candidate correctly expressed the given expression in the form $a + bi$.

In Figure 15, it is also shown that a few candidates (18%) scored low marks. This was due to several factors as follows. Firstly, in part (a) they failed to compare real and imaginary parts in the given equation. One of these candidates for example equated $2x = -12$ instead of $2x - 4y = -12$. This candidate also compared $10y - 4y$ with $-12 + 5$ to get $y = -\frac{7}{6}$ instead of $y = \frac{1}{2}$. Another candidate solved $2x = -12$ to get $y = -6$ instead of solving $2x - 4y = -12$. In part (b), most candidates correctly expressed $(\cos q + i\sin q)^{-n}$ as $\cos(-nq) + i\sin(-nq)$ but could not recognize $\cos(-nq)$ and $\sin(-nq)$ as $\cos(nq)$ and $-\sin(nq)$ respectively. In part (c), a number of candidates failed to substitute $z = x + iy$ into $\frac{z+i}{iz+2}$. Moreover, there were also candidates who substituted $z = 1 + 2i$ into the expression $\frac{z+i}{iz+2}$ but ignored $z = x + iy$. Hence, they obtained incorrect complex numbers such as $1 + 3i$. Extract 14.2 is a sample of a response from one of the candidates who gave incorrect answers in part (c).

use only

4c

$$= \left(\frac{z+i}{2+iz}\right)\left(\frac{2-iz}{2-iz}\right)$$

$$= \frac{(z+i)(2-iz)}{4-(iz)^2}$$

$$= \frac{2z - iz^2 + 2i + z}{4+z^2}$$

$$= \frac{3z + (2-z^2)i}{4+z^2}$$

$$= \frac{3z}{4+z^2} + \frac{(2-z^2)i}{4+z^2}$$

If $z = x+iy$, we get

$$= \frac{3(x+iy) + (2-(x+iy)^2)i}{4+(x+iy)^2}$$

$$= \frac{3x + 3yi + (2 - x^2 + y^2 - 2xyi)i}{4 + (x^2 - y^2 + 2xyi)}$$

$$= \frac{(3x+2xy) + (3y + 2 - x^2 + y^2)i}{4 + (x^2 - y^2 + 2xyi)}$$

c.	When $z = 1+2i$ the resulting complex number is		
	$z + i = (1+2i) + i$		
	$iz + 2 \quad i(1+2i) + 2$		
	$= \frac{1+3i}{i}$		

Extract 14.2: *A sample of an incorrect response to Question 4*

In Extract 14.2, the candidate failed to answer part (c) correctly showing iadequate knowledge on the tested concepts.

2.2.5 Question 5: Trigonometry

The question consisted of parts (a), (b) and (c). In part (a), the candidates were given that A and B are angles of a right angled triangle such that $\cos A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$ and were required to find the value of $\tan 2A$, $\cos(A+B)$ and $\text{cosec}(A-B)$ in the form of $\frac{x}{y}$. In part (b), they were required to: (i) show that $\cot\left(x+\frac{\pi}{2}\right) - \tan\left(x-\frac{\pi}{2}\right) = \frac{2\cos 2x}{\sin 2x}$ and (ii) solve the equation $4\cos 2\theta - 2\cos\theta + 3 = 0$, for $0^0 \leq \theta \leq 360^0$. In part (c), they were required to express $\cos^4\theta$ in terms of cosine of multiples of angle θ.

The analysis of data shows that 64.9 percent of the candidates opted this question whereby 82 percent of the candidates scored from 7.0 to 20.0 marks. Moreover, the data shows that 18 percent of the candidates scored below 7.0 marks, 22 percent scored from 7.0 to 11.5 marks and 60 percent scored from 12.0 to 20.0 marks. Generally, the candidates' performance in this question was good as shown in Figure 16.

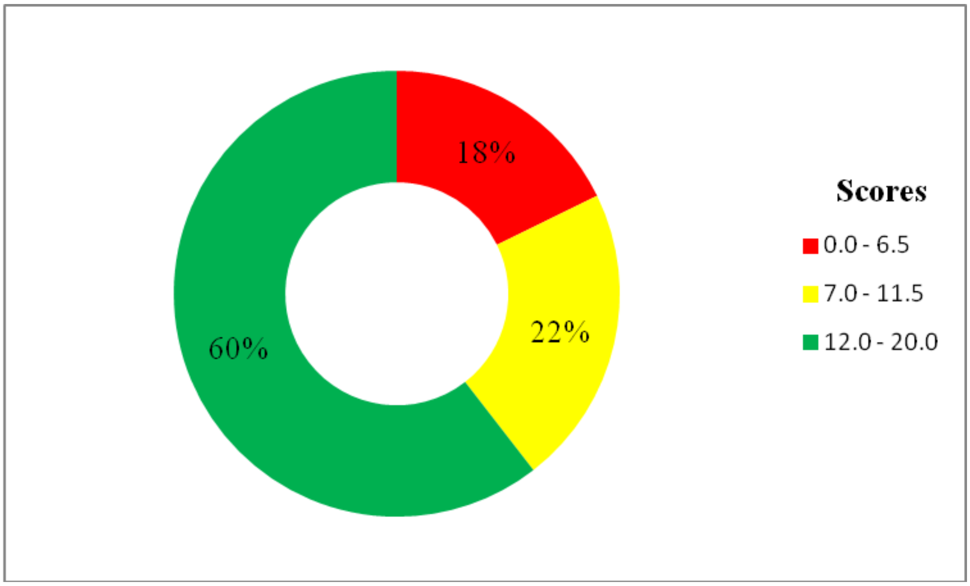


Figure 16: *The candidates' performance in question 5*

The analysis shows that there were 566 candidates (4.4%) who scored 20.0 marks. In part (a), candidates who performed well demonstrated the following competences: Firstly, they correctly used the Pythagoras' Theorem $a^2 + b^2 = c^2$ to find $\sin A = \frac{4}{5}$, $\sin B = \frac{12}{13}$ and $\tan A = \frac{4}{3}$. Secondly, they substituted $\tan A = \frac{4}{3}$ into $\frac{2\tan A}{1-\tan^2 A}$ to get $\tan 2A = -\frac{24}{7}$. Thirdly, they correctly substituted $\sin A = \frac{4}{5}$, $\sin B = \frac{12}{13}$, $\cos A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$ into the compound angle formula for $\cos(A+B)$ to get $\frac{-33}{65}$. Four, they correctly expressed $\operatorname{cosec}(A-B)$ as $\frac{1}{\sin A\cos B - \cos A\sin B}$. Thereafter, they made the appropriate substitutions to get $\operatorname{cosec}(A-B) = -\frac{65}{16}$. In part (b) (i), these candidates correctly replaced $\cot\left(x+\frac{\pi}{2}\right)$ and $\tan\left(x-\frac{\pi}{2}\right)$ with $\frac{\cos\left(x+\frac{\pi}{2}\right)}{\sin\left(x+\frac{\pi}{2}\right)}$ and $\frac{\sin\left(x-\frac{\pi}{2}\right)}{\cos\left(x-\frac{\pi}{2}\right)}$ respectively to get $\dfrac{\cos\left(x+\frac{\pi}{2}\right)}{\sin\left(x+\frac{\pi}{2}\right)}-\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\cos\left(x-\frac{\pi}{2}\right)}$. Thereafter, they changed the resulting equation in order to show that $\cot\left(x+\frac{\pi}{2}\right)-\tan\left(x-\frac{\pi}{2}\right)=\dfrac{2\cos 2x}{\sin 2x}$. In part (b) (ii), they were able to substitute $\cos 2\theta = 2\cos^2\theta - 1$ into $4\cos 2\theta - 2\cos\theta + 3 = 0$ to get $8\cos^2\theta - 2\cos\theta - 1 = 0$. Thereafter, they solved the resulting quadratic equation to get $\theta = 60^\circ$, 104.5°, 255.5° and 300^0. In part (c) (ii), these candidates used correctly the double angle formula to express $\cos^4\theta$ in terms of cosines multiples of θ as $\cos^4\theta = \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$. Extract 15.1 is a sample solution from one of the candidates who had adequate skills on the concepts tested.

5 a) $\cos A = 3/5 = A/H$

but $H^2 = O^2 + A^2$ -- pythagoras theorem

$\therefore H^2 = O^2 + A^2$

$5^2 = O^2 + 3^2$

$\therefore O = 4$

$\therefore \sin A = O/H = 4/5$

$\tan A = O/A = 4/3$.

$\cos B = 5/13 = A/H$

but $H^2 = O^2 + A^2$ -- pythagoras theorem

$\therefore 13^2 = 5^2 + O^2$

$\therefore O = 12$

$\therefore \sin B = O/H = 12/13$

$\tan B = O/A = 12/5$

from $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

$\tan 2A = \dfrac{2(4/3)}{1-(4/3)^2}$

$= -24/7$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\therefore \quad = (3/5)(5/13) - (4/5)(12/13)$

$= -33/65$

$\operatorname{cosec}(A-B) = \dfrac{1}{\sin(A-B)}$

$\sin(A-B) = \sin A\cos B - \sin B\cos A$

$= (4/5)(5/13) - (12/13)(3/5)$

$= -16/65$

$\therefore \operatorname{cosec}(A-B) = -65/16$

$\therefore \tan 2A = -24/7$

$\cos(A+B) = -33/65$

$\operatorname{cosec}(A-B) = -65/16$

5 b) i) $\cot(x + \pi/2) - \tan(x - \pi/2) = \dfrac{2\cos 2x}{\sin 2x}$

proof

$= \tan(\pi/2 - (x + \pi/2)) - \tan(x - \pi/2)$

$= \tan(-x) - \tan(x - \pi/2)$

$= -\tan x - \tan(x - \pi/2)$

$= -\tan x + \tan(\pi/2 - x)$

$= -\tan x + \cot x$

$= \cot x - \tan x$

$= \dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}$

$= \dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x}$

$= \dfrac{\cos 2x}{\frac{1}{2}\sin 2x}$

$= \dfrac{2\cos 2x}{\sin 2x}$

hence shown.

ii) $4\cos 2\theta - 2\cos\theta + 3 = 0$

$4(2\cos^2\theta - 1) - 2\cos\theta + 3 = 0$

$8\cos^2\theta - 4 - 2\cos\theta + 3 = 0$

$8\cos^2\theta - 2\cos\theta - 1 = 0$

$\cos\theta = \dfrac{2 \pm \sqrt{(-2)^2 - 4(-1)(8)}}{2(8)}$

$\cos\theta = \dfrac{2 \pm \sqrt{4 + 32}}{16}$

$\cos\theta = \dfrac{2 \pm 6}{16}$

$\cos\theta = 8/16$ or $-4/16$

$\cos\theta = 1/2$ or $-1/4$

$\theta = \cos^{-1}(1/2)$ or $\cos^{-1}(-1/4)$

5b)(ii)	for $\theta = \cos^{-1}(1/2)$
	$\theta = 60°, 300°$
	for $\theta = \cos^{-1}(-1/4)$
	$\theta = (180° - 75.52°)$ and $(180° + 75.52°)$
	$\theta = 104.48°$ and $255.52°$
	$\therefore \theta = 60°, 104.48°, 255.52°$ and $300°$
c)	$\cos^4\theta = ?$
	from $\cos 2\theta = 2\cos^2\theta - 1$
	and $\cos 4\theta = 2\cos^2 2\theta - 1$, we get
	$\therefore \cos 4\theta = 2(2\cos^2\theta - 1)^2 - 1$
	$\cos 4\theta = 2(4\cos^4\theta + 1 - 4\cos^2\theta) - 1$
	$\cos 4\theta = 8\cos^4\theta + 2 - 8\cos^2\theta - 1$
	$\cos 4\theta = 8\cos^4\theta + 2 - 8\cos^2\theta - 1$
	$8\cos^4\theta = \cos 4\theta - 1 + 8\cos^2\theta$
	$8\cos^4\theta = \cos 4\theta - 1 + 4(2\cos^2\theta)$
	$8\cos^4\theta = \cos 4\theta - 1 + 4(\cos 2\theta + 1)$
	$\cos^4\theta = 1/8(\cos 4\theta - 1 + 4\cos 2\theta + 4)$
	$= 1/8(\cos 4\theta + 4\cos 2\theta + 3)$
	$\therefore \cos^4\theta = 1/8(\cos 4\theta + 4\cos 2\theta + 3)$

Extract 15.1: *A sample of a correct response to Question 5*

In Extract 15.1, the candidate demonstrated competences in applying the appropriate principles and identities to find $\tan 2A$, $\cos(A+B)$ and $\text{cosec}(A-B)$ and solved the trigonometric equation correctly to express $\cos^4\theta$ in terms of cosines multiples of θ.

On the other hand, 22 percent of the candidates scored low marks where in part (a), they failed to write the expansion of $\tan 2A$, $\cos(A+B)$ and $\text{cosec}(A-B)$. The candidates who failed to find $\tan 2A$ wrote the incorrect formulae such as $\tan 2A = \frac{2\tan A}{1+\tan^2 A}$, $\tan 2A = 2\sin A\cos A$ etc. The candidates who failed to calculate $\cos(A+B)$ used the incorrect formulae such as $\cos(A+B) = \text{cosAcosB} + \text{sinAsinB}$ instead of $\cos(A+B) = \text{cosAcosB} - \text{sinAsinB}$. Some candidates who got an incorrect value of $\text{cosec}(A-B)$ quoted $\text{cosec}(A-B)$

as $\text{cosecA} - \text{cosecB} = \frac{1}{\sin A} - \frac{1}{\sin B}$ giving $\frac{1}{6}$ instead of $-\frac{65}{16}$. Other candidates expressed $\text{cosec}(A-B)$ as $\frac{1}{\sin(A+B)} = \frac{1}{\sin A\cos B + \cos A\sin B}$. However, there were candidates who correctly expressed $\text{cosec}(A-B)$ as $\text{cosec}(A-B) = \frac{1}{\sin(A-B)}$ but could not expand $\sin(A-B)$. One of these candidate for example, expressed $\frac{1}{\sin(A-B)}$ as $\frac{1}{\sin A\cos B + \text{cosAsinB}}$ to give $\frac{65}{56}$.

In part (b) (i), these candidates failed to use the compound angle formulae for $\cos(A+B)$, $\cos(A-B)$, $\sin(A+B)$ and $\sin(A-B)$ to show that $\cot\left(x+\frac{\pi}{2}\right) - \tan\left(x-\frac{\pi}{2}\right) = \frac{2\cos 2x}{\sin 2x}$. Other candidates expressed the left hand side of the given equation as $\frac{1}{\tan\left(x+\frac{\pi}{2}\right)} - \tan\left(x - \tan\frac{\pi}{2}\right)$. Hence, they could not prove the correctness of the given trigonometric equation. In part (b) (ii), the candidates failed to solve the value of θ in the given range because they were unable to expand $\cos 2\theta$. One candidate for example substituted $\cos 2\theta = \sin^2\theta + \cos^2\theta$ into the equation $4\cos 2\theta - 2\cos\theta + 3 = 0$ hence got $\theta = \cos^{-1}\left(\frac{7}{2}\right)$ instead of $\theta = 60^\circ,\ 104.5^\circ,\ 255.5^\circ,\ 300$. The candidate was supposed to understand that the trigonometric identity $\sin^2\theta + \cos^2\theta$ gives 1 and not $\cos 2\theta$.

In part (c), most of these candidates failed to express $\cos^4 q$ in terms of cosines multiples of θ. For instance, one of these candidates substituted incorrectly $\cos^3 q = \frac{1}{4}(\cos 3q + 3\cos q)$, into $\cos q(\cos^3 q)$ to get the equation

$\cos^4 q = \frac{1}{4}\cos q \cos 3q + 3\cos^2 q$ instead of $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. Another candidate in this category expressed $\cos^4\theta$ as $(\cos 2\theta - \sin 2\theta)(\cos 2\theta - \sin 2\theta)$ and simplified it to get $2(1 - \sin 4\theta)$. Extract 15.2 illustrates an incorrect solution from one of the candidates who had inadequate knowledge of trigonometric concepts.

5 (a) $\cos(A+B) = \cos A \cos B + \sin A \sin B$

using the obtained values

$\cos(A+B)$

$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{13}\right)$

$= \frac{3}{13} + \frac{48}{65}$

$= \frac{63}{65}$

$\therefore \cos(A+B) = \frac{63}{65}$

$\operatorname{cosec}(A-B) = \frac{1}{\sin(A+B)}$

$\operatorname{cosec}(A-B) = \frac{1}{\sin A \cos B + \cos A \sin B}$

$= \frac{1}{\left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{12}{13}\right)}$

$= \frac{1}{\frac{4}{13} + \frac{36}{65}}$

$= \frac{1}{\frac{56}{65}}$

$= \frac{65}{56}$

$\therefore \operatorname{cosec}(A-B) = \frac{65}{56}$.

Extract 15.2: *A sample of an incorrect response to Question 5 (a)*

In Extract 15.2, the candidate wrote inappropriate expansions of $\cos(A+B)$ and $\operatorname{cosec}(A-B)$ showing inadequate knowledge of the tested concept.

2.2.6 Question 6: Algebra

The question was

(a) By using the first five terms in the expansion of $(1+x)^n$, find the value of $(1.98)^{10}$ correct to three decimal places.

(b) The polynomial function $p(x)=x^5+4x^2+ax+b$ leaves the remainder of $2x+3$ when it is divided by x^2-1. Use the remainder theorem to find the values of *a* and *b*.

(c) The roots of the equation $x^2+2mx+n=0$ differ by 2. Show that $m^2=1+n$.

(d) If $A=\begin{pmatrix} 4 & -1 & 1 \\ 0 & 0 & 2 \\ m & -1 & 1 \end{pmatrix}$ is a singular matrix, find the value of *m*.

(e) Use Crammer's rule to solve the system of equations $\begin{cases} 5x+6y+4z=5 \\ 7x-4y-3z=8 \\ 2x+3y+2z=2 \end{cases}$.

The analysis of data shows that the question was opted by 9129 candidates (71.5%) and 3642 candidates (28.5%) left it unanswered. Further analysis shows that 68.1 percent of the candidates scored at least 7.0 marks and 38.7 percent scored from 12.0 to 20.0 marks. Generally this question was well performed as summarized in Figure 17.

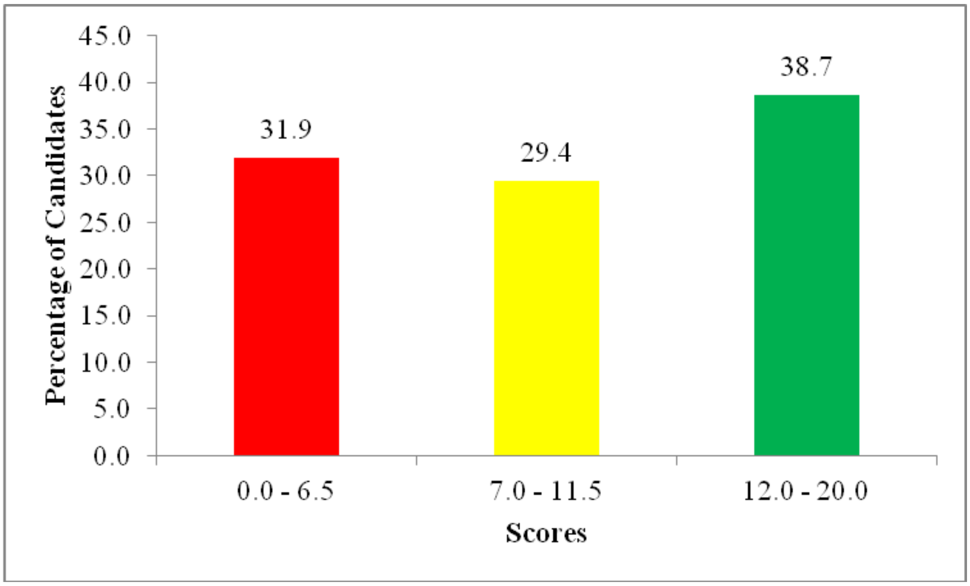


Figure 17: *Candidates' performance in question 6*

A good number of candidates (38.7%) out of those who chose to attempt this question scored high marks. In part (a), they were able to expand $(1+x)^n$ into $1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\frac{n(n-1)(n-2)(n-3)}{4!}x^4+\ldots$. Then, they rearranged $(1.98)^{10}$ as $(2(1-0.01))^{10}$ where $x=-0.01$ and $n=10$. Thereafter, they correctly substituted the values of x and n into

$$2^{10}\left[1+10(-0.01)+\frac{10\times 9}{2}(-0.01)^2+\frac{10\times 9\times 8}{6}(-0.01)^3+\frac{10\times 9\times 8\times 7}{24}(-0.01)^4\right]$$

which gave 926.087.

In part (b), the candidates were able to find the values of a and b through the following steps: one, they equated the divisor x^2-1 to zero to get -1 and 1; two, they correctly used the reminder theorem to formulate the equations $a+b=0$ and $a-b=2$; three, they solved the equation in step two to obtain $a=1$ and $b=-1$.

The candidates who attempted part (c) correctly demonstrated the following competences: one, they managed to establish the required roots which differ by 2 as a and $(a-2)$; two, they used the quadratic equation $x^2+2mx+n=0$

together with the roots in step 1 to formulate the equations $a^2 - 2a = n$ and $a + a - 2 = 2m$; three, they solved the equations in step two to confirm the correctness of $m^2 = 1 + n$.

In part (d), they equated the determinant of matrix A to zero to obtain $4\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ m & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ m & -1 \end{vmatrix} = 0$. Thereafter, they correctly computed the value of m to get $m = 4$.

In part (e), the candidates demonstrated the following strengths: One, they computed $\begin{vmatrix} 5 & 6 & 4 \\ 7 & -4 & -3 \\ 2 & 3 & 2 \end{vmatrix}$ to get the determinant of the coefficient matrix equal to 1; Two, they replaced the x-values in the first column of the coefficient matrix with the values after the equal sign to get the matrix $\begin{pmatrix} 5 & 6 & 4 \\ 8 & -4 & -3 \\ 2 & 3 & 2 \end{pmatrix}$; three they calculated the determinant of the matrix in step 2 to get 1; four, they replaced the y-values in the second column of the coefficient matrix with the values after the equal sign to get the matrix $\begin{pmatrix} 5 & 5 & 4 \\ 7 & 8 & -3 \\ 2 & 2 & 2 \end{pmatrix}$; five, they computed the determinant of the matrix in step 4 to get 2; six, they replaced the z-values in the third column of the coefficient matrix with the values after the equal sign to get the matrix $\begin{pmatrix} 5 & 6 & 5 \\ 7 & -4 & 8 \\ 2 & 3 & 2 \end{pmatrix}$; seven, they computed the determinant in step six to get -3. Eight, they correctly divided the determinants obtained in step two, five and seven by the determinant of the coefficient matrix to get $x = 1$, $y = 2$ and $z = -3$ which was the required solution to the given system of equations. Extract 16.1 is a sample of responses from a candidate who correctly answered this question.

6\. (a) $(1+x)^n$

From

$$(1+x)^n = 1 + nx + \frac{n\cdot(n-1)x^2}{2!} + \frac{n\cdot(n-1)(n-2)x^3}{3!} + \frac{n\cdot(n-1)(n-2)(n-3)x^4}{4!}$$

Required expansion of $(1\cdot98)^{10}$.

$$(1\cdot98)^{10} = (2-0\cdot02)^{10}$$

$$= \left[2\left(1-\frac{0\cdot02}{2}\right)\right]^{10}$$

$$= 2^{10}(1-0\cdot01)^{10}$$

$$2^{10}(1-0\cdot01)^{10} = 2^{10}\left[1 + 10\cdot(-0\cdot01) + \frac{10\cdot(9)(-0\cdot01)^2}{2!} + \frac{10\cdot9\cdot8(-0\cdot01)^3}{3!} + \frac{10\cdot9\cdot8\cdot7\cdot(-0\cdot01)^4}{4!}\right]$$

$$2^{10}(1-0\cdot01)^{10} = 2^{10}\left[1 - 0\cdot1 + 4\cdot5\times10^{-3} - 1\cdot2\times10^{-4} + 2\cdot1\times10^{-6}\right]$$

$$= 2^{10}(0\cdot9043821)$$

$$= 926\cdot0872448$$

Correct to three decimal places.

$926\cdot0872448 \approx 926\cdot087$

$\therefore (1\cdot98)^{10} = 926\cdot087$

(b). Given the polynomial

$x^5 + 4x^2 + ax + b$

remainder $2x+3$ when divided by x^2-1.

6- (b). soln.

$$P(x^2-1) = x^5 + 4x^2 + ax + b = 2x+3.$$

$$P(\pm1) = x^5 + 4x^2 + ax + b = 2x+3.$$

start with positive 1.

$$P(1) = (1)^5 + 4(1)^2 + a(1) + b = 2(1)+3$$

$$1 + 4 + a + b = 2+3$$

$$a+b+5 = 5$$

$$a+b = 0 \quad \text{------} \quad (i)$$

then negative -1.

$p(-1) = (-1)^3 + 4(-1)^2 + a(-1) + b = 2(-) + 3$

$-1 + 4 - a + b = -2 + 3$

$-a + b + 3 = 1.$

$-a + b = -2. \quad \text{-----} \quad (ii)$

Solve simultaneously the two eqn.

$a + b = 0$

$-a + b = -2.$

Hence $a = 1$ and $b = -1$.

∴ The value of a and b are 1 and -1

(c). Given the quadratic equation.

$x^2 + 2mx + n = 0.$

roots.

Sum of roots $= -b/a$

product of roots $= c/a$

6. (c) ~~Sum~~ $\alpha + \beta = -2m \quad \text{----} \quad (i)$

$\alpha\beta = n. \quad \text{----} \quad (ii)$

but the roots differ by 2.

$\alpha - \beta = 2. \quad \text{----} \quad (iii)$

$\alpha = 2 + \beta \quad \text{------} \quad (iv)$

Substitute eqn (iv) into eqn (i)

$2 + \beta + \beta = -2m$

$2 + 2\beta = -2m.$

$2\beta = -2m - 2.$

$\beta = -m - 1. \quad \text{----} \quad (v)$

then.

substitute eq (v) into eqn (iv)

$\alpha = 2 + (-m - 1)$

$\alpha = 2 - m - 1$

$\alpha = 1 - m \quad \text{----} \quad (vi)$

~~Sol~~ From.

$\alpha\beta = n.$

substitute the required information

$(1-m)(-m-1) = n.$

$(-m-1+m^2+m) = n$

$-1+m^2 = n$

$m^2 = 1+n$

$m^2 = 1+n$

Hence shown.

use only

6 (d). Given,

$$A = \begin{pmatrix} 4 & -1 & 1 \\ 0 & 0 & 2 \\ m & -1 & 1 \end{pmatrix} \text{ is singular}$$

Since we know singular matrix is the one whose determinant is equal to zero.

hence, we find determinant of element A.

$$\det A = 4\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} - -1\begin{vmatrix} 0 & 2 \\ m & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ m & -1 \end{vmatrix}$$

$\det A = 8 - 2m + 0$

but

$\det A = 0$

$0 = 8 - 2m$

$m = 4.$

$\therefore$ The value of m is 4.

(e). soln.

By using cramer's rule.

Given.

$5x + 6y + 4z = 5$

$7x - 4y - 3z = 8$

$2x + 3y + 2z = 2.$

Matrix form.

$$\begin{pmatrix} 5 & 6 & 4 \\ 7 & -4 & -3 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 2 \end{pmatrix}$$

6. (e).

$$\begin{pmatrix} 5 & 6 & 4 \\ 7 & -4 & -3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 2 \end{pmatrix}$$

By cramer's rule.

$$\det(D) = 5\begin{vmatrix} -4 & -3 \\ 3 & 2 \end{vmatrix} - 6\begin{vmatrix} 7 & -3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 7 & -4 \\ 2 & 3 \end{vmatrix}$$

$$= 5 - 120 + 116$$

$$= 1.$$

about X.

~~det (δx)~~ $= \begin{pmatrix} 5 & 6 & 4 \\ 8 & -4 & -3 \\ 2 & 3 & 2 \end{pmatrix}$

$$\det(\delta x) = 5\begin{vmatrix} -4 & -3 \\ 3 & 2 \end{vmatrix} - 6\begin{vmatrix} 8 & -3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 8 & -4 \\ 2 & 3 \end{vmatrix}$$

$$\det(\delta x) = 1.$$

about Y.

$$\begin{pmatrix} 5 & 5 & 4 \\ 7 & 8 & -3 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\det(\delta Y) = 5\begin{vmatrix} 8 & -3 \\ 2 & 2 \end{vmatrix} - 5\begin{vmatrix} 7 & -3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 7 & 8 \\ 2 & 2 \end{vmatrix}.$$

$$\det(\delta Y) = 2.$$

about $Z = \begin{pmatrix} 5 & 6 & 5 \\ 7 & -4 & 8 \\ 2 & 3 & 2 \end{pmatrix}$

$$\det(\delta Z) = 5\begin{vmatrix} -4 & 8 \\ 3 & 2 \end{vmatrix} - 6\begin{vmatrix} 7 & 8 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 7 & -4 \\ 2 & 3 \end{vmatrix}$$

$$\det(\delta Z) = -3.$$

6. (e) then.

$$X = \frac{\delta X}{\delta} = \frac{1}{1}$$

$x = 1$

$$Y = \frac{\delta Y}{\delta} = \frac{2}{1}$$

$Y = 2$

$$Z = \frac{\delta Z}{\delta} = \frac{-3}{1}$$

$z = -3.$

∴ The value of x, y and z are (1, 2, -3)

Extract 16.1: *A correct response to Question 6.*

In Extract 16.1, the candidate was able to use binomial theorem in the expansion of $(1+x)^n$ to approximate the value of $(1.98)^{10}$. The candidate was also able to use remainder theorem to get the values of a and b. Furthermore, the candidate applied the concept of the roots of equation to show that $m^2 = 1 + n$. Finally, the candidate used the condition of singular matrix and Crammers rule to answer part (d) and (e).

Despite the good performance in question 6, some candidates (29.4%) had an average performance in this question. These candidates responded correctly to part (b), (d) and (e) whereas in part (a) and (c) they produced incorrect responses.

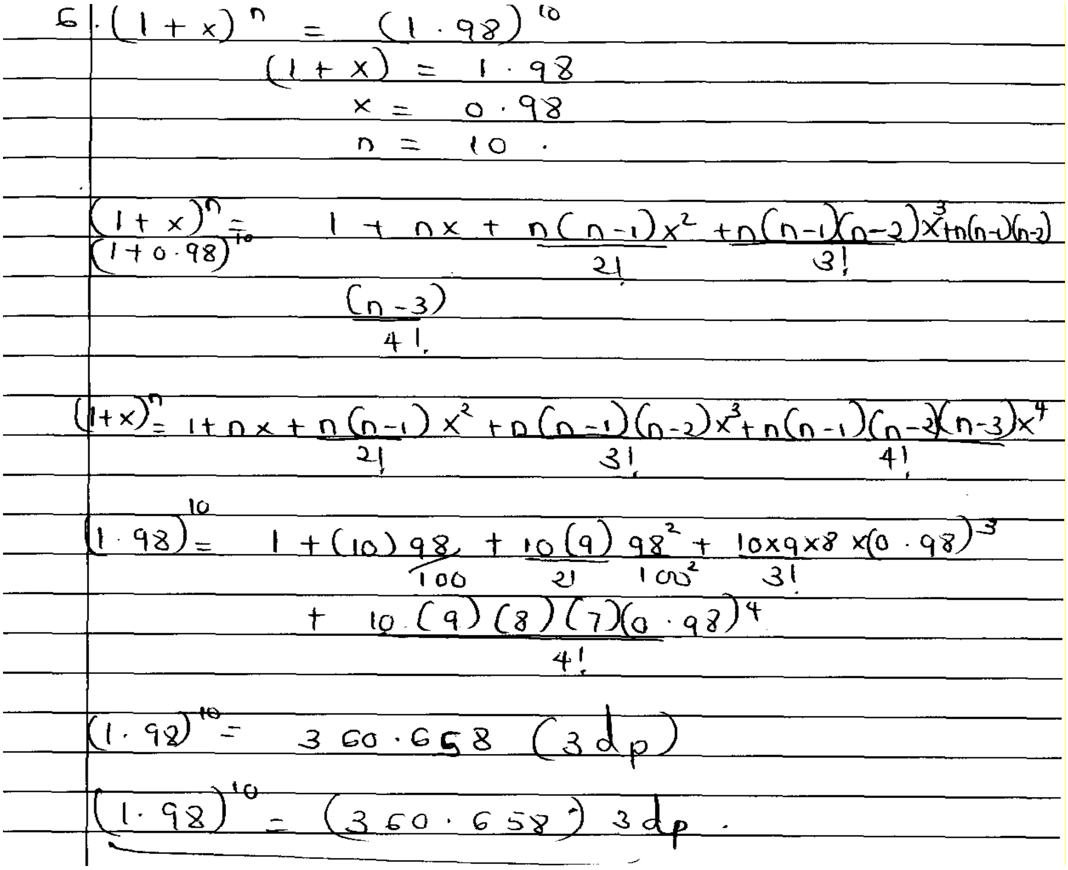
Additionally, there were several candidates (31.9%) who scored low marks. The candidate's weak performance was due to several reasons. In part (a), some candidates had inadequate knowledge on binomial expansion of $(1+x)^n$. For example, a number of candidates quoted $(1+x)^n$ as $\sum_{r=0}^{n} {}^{n}c_r a^{n-r} b^r$. These candidates were supposed to understand that $\sum_{r=0}^{n} {}^{n}c_r a^{n-r} b^r$ is the expansion of $(a+b)^n$. Other candidates in this category used the Pascal's triangle in the expansion of $(1+x)^5$ to get $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$ which did not help them in finding the approximation $(1.98)^{10}$. A considerable number of candidates also quoted $(1.98)^{10}$ as $(1+0.98)^{10}$. Due this reason, they substituted $x = 0.98$ into $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$ to get the value of $(1.98)^{10}$ equal to 360.658 instead of 926.087. There were also candidates who were unable to do the computation correctly. The candidates in this category ended up getting incorrect answers like 1121.671.

In part (b), the candidates lacked skills of using remainder theorem to answer the question. Some of these candidates for example, divided $x^2 - 1$ by $2x+3$ to get $\frac{1}{2}x+\frac{3}{4}$ and then compared them by writing $ax+b=\frac{1}{2}x+\frac{3}{4}$. As a result, they incorrectly concluded that $a=\frac{1}{2}$ and $b=\frac{3}{4}$. Other candidates in this category solved the equation $x^2 - 1= 0$ to get $x = 1$ and $x = -1$. Thereafter, they equated $f(1)$ with $f(-1)$ to get $a = -1$ and $b = -1$. Though the value of b was accidentally correct the value of a was not correct.

In part (c), most of these candidates recognised the roots of the given quadratic equation as a and $(a + 2)$ instead of a and $(a - 2)$ thus they failed to show that $m^2 = 1+ n$. There were also candidates who could not equate the sum and product of the roots a and $(a - 2)$ to $-2m$ and n respectively. For instance, one of the candidates manipulated the ratio $\frac{a+b}{ab} = -\frac{2m}{n}$ to get $\frac{2b+2}{b^2+2b} = -\frac{2m}{n}$. This technique did not help the candidate to show that $m^2 = 1+ n$.

In part (d), the candidates failed to understand that every singular matrix has the determinant of zero. There were candidates for instance, who computed $|A| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 0 & 2 \\ m & -1 & 1 \end{vmatrix} = 1$ to get $m=\frac{7}{2}$ instead of $m=\frac{1}{4}$. Other candidates in this category computed $|A| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 0 & 2 \\ m & -1 & 1 \end{vmatrix} \neq 0$ to give $m \neq 4$ which was not correct.

In part (e), most candidates used the pattern $\begin{pmatrix} - & + & - \\ + & - & + \\ - & + & - \end{pmatrix}$ instead of $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ to find the determinants D, D_x, D_y and D_z. These candidates ended up getting incorrect values of x, y and z such as $x = 182.4$, $y = -127$ and $z = 3046$. Extract 16.2 is a sample of incorrect responses showing some mistakes which were done by the candidates.



6. $(1+x)^n = (1.98)^{10}$

$(1+x) = 1.98$

$x = 0.98$

$n = 10$.

$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)}{4!}$

$(1+0.98)^{10}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4$

$(1.98)^{10} = 1 + (10)\frac{98}{100} + \frac{10(9)}{2!}\frac{98^2}{100^2} + \frac{10\times9\times8}{3!}\times(0.98)^3$

$+ \frac{10(9)(8)(7)(0.98)^4}{4!}$

$(1.98)^{10} = 360.658\ (3dp)$

$(1.98)^{10} = (360.658)\ 3dp$.

Extract 16.2: *A sample of an incorrect response to Question 6*

2.2.7 Question 7: Differential Equations

The question was

(a) Form a differential equation whose solution is $x = \tan(Ay)$.

(b) Solve the differential equation $\frac{d^2q}{dt^2} - 4\frac{dq}{dt} + 4q = \frac{3}{7}$.

(c) A biologist is researching the population of specie. She tries a number of different models for the rate of growth of the population and solves them to compare with observed data. Her first model is $\frac{dn}{dt} = kn\left(1 - \frac{n}{a}\right)$, where n is the population at time t years, k is a constant and a is the maximum population sustainable by the environment. Given that $k = 0.2$, $a = 100,000$ and the initial population is $30,0000$;

(i) find the general solution of the differential equation.

(ii) estimate the population after 5 years to 2 significant figures.

This question was opted by a few candidates (21.3%), whereby, 68.1 percent of the candidates passed. Further analysis shows that 31.9 percent of the candidates scored from 0.0 to 6.5 marks, 37.7 percent scored from 7.0 to 11.5 marks and 30.4 percent scored from 12.0 to 20.0 marks. Moreover, the analysis shows that 125 candidates (4.6%) scored all 20.0 marks and 140 candidates (5.1%) scored 0. Generally, candidates performed well in this question as summarized in Figure18.

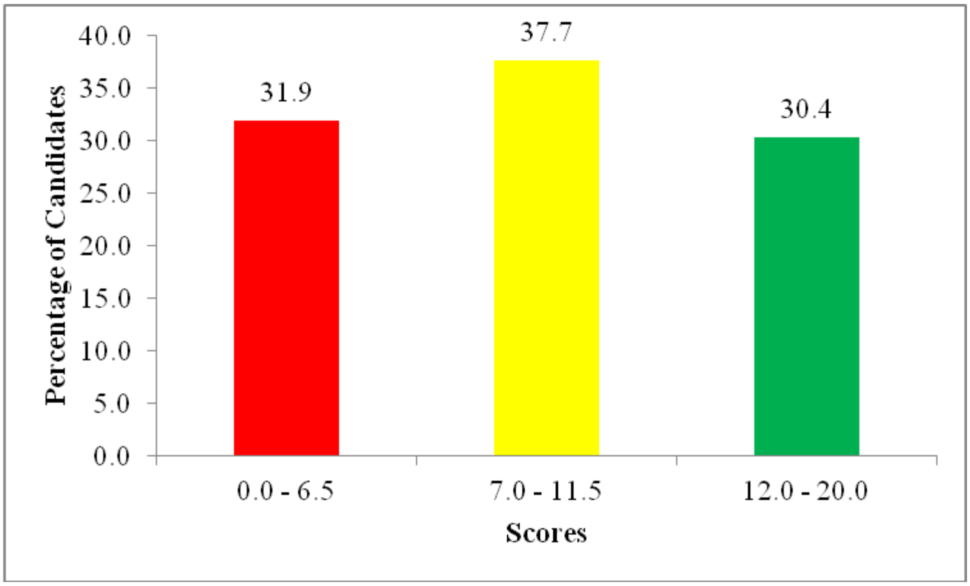


Figure 18: *Candidates' performance in question 7*

As shown in Figure 18, several candidates (25.4%) scored high marks. The analysis of the candidates' responses revealed that, they performed well in this question. This good performance was due to several factors: In part (a), the candidates were able to formulate the differential equation from $x = \tan(Ay)$ by differentiating x with respect to y i.e. $\frac{dx}{dy} = A\sec^2(Ay)$, then inserted $A = \frac{1}{y}\tan^{-1}x$ in $\frac{dx}{dy} = A\sec^2(Ay)$ to get $\frac{dx}{dy} = \frac{(1+x^2)}{y}\tan^{-1}x$ as per the requirements of the question.

In part (b), the candidates found the correct complementary solution as $q_1 = (At+B)e^{2t}$ and the particular solution as $\theta_2 = \frac{3}{28}$. Thereafter, they added them to get $q = (At+B)e^{2t} + \frac{3}{28}$ as the general solution.

In part (c) (i), they managed to separate the variables of the differential equation $\frac{dn}{dt} = kn\left(1-\frac{n}{a}\right)$ as $\int \frac{a}{n(a-n)}dn = \int kdt$. They solved it by using the initial values to obtain $n = \frac{3ae^{kt}}{7+3e^{kt}}$. In part (c) (ii), they correctly substituted $k = 0.2$, $a = 100{,}000$, $t = 5$ years into $n = \frac{3ae^{kt}}{7+3e^{kt}}$ so as to estimate the population (*n*) as 54000 in two significant figures. . Extract 17.1, is a sample of correct responses from one of the candidates who answered this question.

7. a) $x = \tan(Ay)$

$\tan^{-1} x = Ay$

$A = \frac{1}{y}\tan^{-1}x$

$A = \frac{\tan^{-1}x}{y}$

differentiate both sides.

$\frac{d}{dx}\left(\frac{\tan^{-1}x}{y}\right) = \frac{d(A)}{dx}$

$\frac{y\frac{d}{dx}(\tan^{-1}x) - \tan^{-1}x \cdot \frac{dy}{dx}}{y^2} = 0$

$y\frac{d}{dx}(\tan^{-1}x) - \tan^{-1}x \cdot \frac{dy}{dx} = 0$

but $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$

$\frac{y}{1+x^2} - \frac{dy}{dx}\tan^{-1}x = 0$

$\therefore (\tan^{-1}x)\frac{dy}{dx} = \frac{y}{1+x^2}$

7 b) given D.E

$\frac{d^2\theta}{dt^2} - 4\frac{d\theta}{dt} + 4\theta = 3/7$

for complementary function (θ_c)

when; $\frac{d^2\theta}{dt^2} - 4\frac{d\theta}{dt} + 4\theta = 0$ —1

let $\theta_c = e^{mt}$

$\theta_c' = me^{mt}$

$\theta_c'' = m^2e^{mt}$

substitute in the D.E

$m^2e^{mt} - 4(me^{mt}) + 4e^{mt} = 0$

$e^{mt}(m^2-4m+4) = 0$

$e^{mt} \neq 0$, $m^2-4m+4=0$ (A.Q.E)

$m^2-4m+4=0$

$m=2$

$\therefore \theta_c = e^{2t}(At+B)$

for particular Integral (θ_p)

let $\theta_p = c$

$\theta_p' = 0$

$\theta_p'' = 0$

substitute in the D.E

$0 - 4(0) + 4c = 3/7$

$4c = 3/7$

$c = 3/28$

$\therefore \theta_p = 3/28$

for general solution,

$\theta = \theta_c + \theta_p$

$\therefore \theta = e^{2t}(At+B) + 3/28$

7. c) i) given

$\frac{dn}{dt} = kn(1-n/a)$

$\frac{dn}{n(1-n/a)} = k\,dt$

Integrate both sides

$$\int_{n_0}^{n} \frac{dn}{n(1-n/a)} = \int_0^t k\,dt$$

$$\int_{n_0}^{n} \frac{dn}{n(1-n/a)} = kt$$

consider

$$\frac{1}{n(1-n/a)} = \frac{A}{n} + \frac{B}{1-n/a}$$

$$1 = A(1-n/a) + Bn$$

when $n=0$

7. c) i) $A=1$

when $n=a$

$$1 = A(1-a/a) + aB$$

$$B = 1/a$$

$$\frac{1}{n(1-n/a)} = \frac{1}{n} + \frac{1}{a(1-n/a)}$$

$$\frac{1}{n(1-n/a)} = \frac{1}{n} + \frac{1}{a-n}$$

$$\int \frac{dn}{n(1-n/a)} = \int \frac{dn}{n} + \int \frac{dn}{a-n} \quad \text{(without limits)}$$

$$\int \frac{dn}{n(1-n/a)} = \ln(n) + -\ln(a-n)$$

with limits

$$= \ln n + -\ln(a-n) \Big|_{n_0}^{n}$$

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7. c) i) $\ln n - \ln(a-n) \Big|_{n_0}^{n} = kt$

$$[\ln n - \ln(a-n)] - [\ln n_0 - \ln(a-n_0)] = kt$$

$$\ln\left(\frac{n}{a-n}\right) - \ln\left(\frac{n_0}{a-n_0}\right) = kt$$

$$\ln\left(\frac{n}{a-n} \times \frac{a-n_0}{n_0}\right) = kt$$

given $a = 100000$

$n_0 = 30000$

$k = 0.2$

$$\ln\left(\frac{n}{100000-n} \times \frac{100000-30000}{30000}\right) = 0.2t$$

$$\ln\left(\frac{7n}{300000-3n}\right) = 0.2t$$

General solution of the differential equation

$\therefore \ln\left(\frac{7n}{300000-3n}\right) = 0.2t$

7. c) ii) Population after $t = 5$ years

from, D.E

$\ln\left(\frac{7n}{300000-3n}\right) = 0.2t$

$\frac{7n}{300000-3n} = e^{0.2(5)}$

$7n = 2.718(300000-3n)$

$15.155n = 2.718 \times 300000$

$n = 53810.15262$

$\therefore$ population after 5 years;

$n \approx 54000$

Extract 17.1: *A sample of a correct response to Question 7*

In Extract 17.1, the candidate was able to formulate the differential equation from the given general solution and solve correctly the second order differential equation. Moreover, the candidate applied the concepts of first order differential equation in solving the real life problem in part (c).

On the other hand, a number of candidates had average performance in this question. These candidates formulated correctly the required differential equation in part (a) and solved the given differential equation in part (b). However, they did not apply differential equations in solving a real life problem in part (c).

A few candidates (31.9%) scored low marks in this question. The analysis of the candidates' responses shows that, they had insufficient knowledge and skills of differentiation and integration techniques. In part (a), they failed to realize that the order of the differential equation is determined by the number of arbitrary constants available in the given general solution. The candidates ended up differentiating the given function twice and obtained incorrect second order differential equations such as $\frac{d^2y}{dx^2} = \left[\frac{2x}{1+x^2}\right]\left(\frac{dy}{dx}\right)^2$ and

$\left[1+x^2\right]\tan^{-1}x\frac{d^2y}{dx^2}+\frac{2xy}{1+x^2}=0$. Some of them did not take trouble to eliminate the constant of integration as they presented $\frac{dx}{dy} = A\sec^2(Ay)$ as the required differential equation. Other candidates in this category had insufficient knowledge of trigonometry as well as differential equation. One of these candidates for example, wrote $\tan A=\frac{x}{y}$ and $\frac{1}{\tan A}\frac{d^2y}{dx^2}=-2x\sec^2 A+\frac{1}{\sin A\cos A}-\frac{\cos A}{\sin A}$ as the formulated differential equation which were incorrect responses.

In part (b), most of the candidates failed to formulate the auxiliary quadratic equation $m^2 - 4m + 4 = 0$. For example, a number of candidates quoted it as $x^2-4x+4=\frac{3}{7}$. This mistake prevented them from getting the complementary solution. There were also candidates who used the variables x and y instead of θ and t. Due to this reason, they ended up getting $y=(Ax+B)e^{2x}+\frac{3}{7}$ instead of $q = (At + B)e^{2t} + \frac{3}{28}$. In some cases, a number of candidates gave incorrect standard complementary and particular functions as $y=(A+B)e^{\alpha t}$ and $\theta=Ae^{\lambda t}$ respectively. They ended up giving incorrect solutions like $y = (A + B)e^{2t} + \frac{3}{7}e^{l\,t}$. Furthermore, a considerable number of candidates formulated correctly the auxiliary quadratic equation as $x^2-4x+4=0$ but got wrong values for x such as $x=\pm 2$ instead of $x=2$.

In part (c) (i), it was noted that several candidates failed to separate the variables of the given differential equation to get $\frac{a}{n(a-n)}dn=kdt$. These candidates ended up writing incorrect equations like $\frac{dn}{dt}=k\frac{dn}{dt}$, $kat=\frac{1}{n}\ln\left(\frac{n}{a-n}\right)$, etc. which gave wrong solution of the given differential equation. It was further noted that a number of candidates substituted the given initial values in the equation $\frac{dn}{dt} = kn\left(1 - \frac{n}{a}\right)$ to get $\frac{dn}{dt} = 4200$. This was also incorrect. Extract 17.2 shows a sample of responses from one of the candidates who failed to answer this question correctly.

7(a) $x = \tan(AY)$

Now

$dx/dy = A\sec^2(AY)$

$dx/dy = A(1 + \tan^2(AY).)$

$dx/dy = A(1 + x^2).$

$dx/dy = A + Ax^2$

$d^2x/dy^2 = 2Ax\, dx/dy$ —— (1)

But

from $dx/dy = A(1+x^2).$

$A = \left(\frac{1}{1+x^2}\right) dx/dy$

Then from Equation (1)

$d^2x/dy^2 = 2\left(\frac{1}{1+x^2}\right)\frac{dx}{dy} \cdot x\, dx/dy$

$d^2x/dy^2 = \frac{2x}{1+x^2}\left(dx/dy\right)^2.$

Then the differential Equation is given by

$d^2x/dy^2 = \left(\frac{2x}{1+x^2}\right)\left(\frac{dx}{dy}\right)^2.$

Extract 17.2: *A sample of an incorrect response to Question 7 (a)*

In Extract 17.2, the candidate gave the formula for the second order differential equation instead of first order differential equation in part (a).

2.2.8 Question 8: Coordinate Geometry II

The question was

(a) Express $x^2 + y^2 = 2x + 2y$ in polar form.

(b) Find the equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ joining the points whose eccentric angles are θ and ϕ.

(c) Show that the point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Hence, find the equation of the tangent on the given hyperbola at point P.

(d) Show whether the equation of the normal to the parabola $y^2 = 4ax$ at point (x_1, y_1) is $(x - x_1)y_1 + 2a(y - y_1) = 0$.

(e) (i) Change the polar equation $r^2(b^2\cos^2 q + a^2\sin^2 q) = a^2b^2$ into Cartesian equation .

(ii) Draw the graph of $r = 2(1 + \cos\theta)$.

Approximately 5,209 candidates (40.8%) opted to attempted this question, whereby 11 percent scored from 0 to 6.5 marks, 23 percent from 7.0 to 11.5 marks and 66.0 percent from 12.0 to 20.0. The candidates' performance in this question was good, as 89 percent of the candidates scored at least 7.0 marks. Figure 19 is a summary of the candidates' performance in this question.

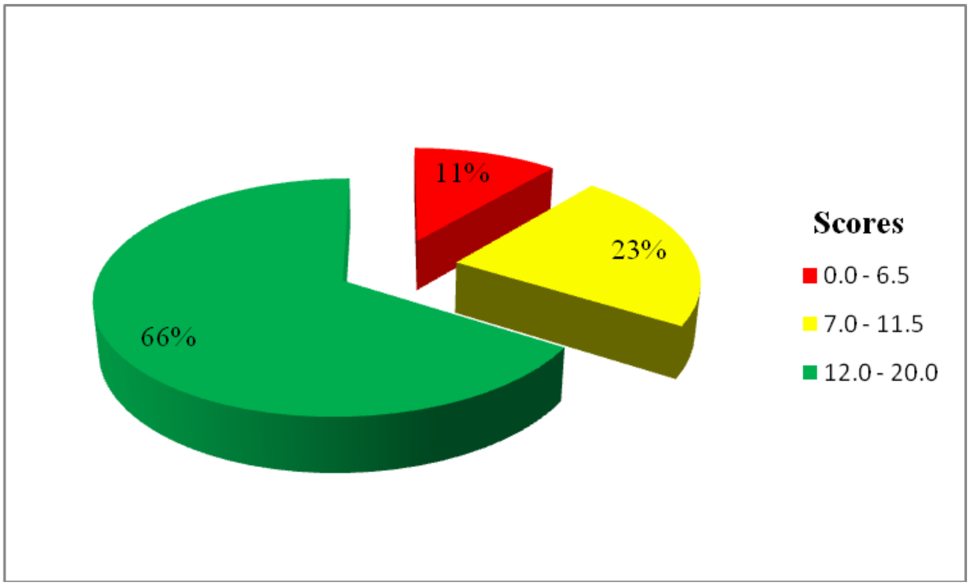


Figure 19: *Candidates' performance in question 8*

The analysis of data shows that, there were 128 candidates who scored full marks. These candidates had adequate knowledge of Coordinate Geometry II. In part (a), they correctly substituted $x = r\cos\theta$ and $y = r\sin\theta$ into the cartesian equation $x^2 + y^2 = 2x + 2y$ to get $r = 2(\cos\theta + \sin\theta)$ which was the required solution.

In part (b), the candidates used correctly the eccentric angles θ and ϕ to identify the end points of the chord as $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$. Then, they used the formula $y = m(x - x_0) + y_0$ and factor formula to find the equation of chord as $bx\cos\frac{1}{2}(\theta+\phi) + ya\sin\frac{1}{2}(\theta+\phi) - ab\cos(\theta-\phi) = 0$.

In part (c), the candidates were able to do the following: one, they substituted the parametric equation $x = a\sec\theta$ and $y = b\tan\theta$ into the expression $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ to get 1. Hence, they were able to show that the point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; two, they differentiated the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to x to get the slope $\frac{dy}{dx}=\frac{b^2x}{a^2y}$; three, they substituted the coordinates of point $P(a\sec\theta, b\tan\theta)$ into the slope in step two to get $\frac{dy}{dx}=\frac{b\sec\theta}{a\tan\theta}$ and four, they correctly substituted the slope obtained in step three and the coordinates of point P into the formula $y= m(x- x_0)+ y_0$ to get the equation of a tangent to the hyperbola equal to $bx\sec\theta - ay\tan\theta = ab$.

In part (d), the candidates were able to show whether the equation of a normal to the parabola $y^2=4ax$ at point (x_1, y_1) is $(x-x_1)y_1+2a(y-y_1)=0$ through the following five main steps: one, they differentiated the equation $y^2=4ax$ at point (x_1, y_1) to get the slope of the tangent to the parabola equal to $\frac{2a}{y_1}$; two, they recognized that the tangent and normal to the parabola are perpendicular to each other. Thus, they correctly substituted the gradient (m_1) obtained in step one into the formula $m_1m_2=-1$ to obtain the gradient (m_2) of a normal to the given parabola equal to $\frac{-y_1}{2a}$; three, they used the gradient in step two together with the formula $y= m(x- x_0)+ y_0$ and the coordinates of point (x_1, y_1) to confirm that $(x-x_1)y_1+2a(y-y_1)=0$ is the equation of a normal line to the given parabola.

In part (e) (i), they correctly substituted $x=r\cos\theta$ and $y=r\sin\theta$ into $r^2b^2\cos^2 q+ r^2a^2\sin^2 q= a^2b^2$ to get the corresponding cartesian equation equal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. In part (e) (ii), these candidates managed to prepare a table of values for $r=2(1+\cos\theta)$ and q from $q= 0°$ to $q= 360°$. Thereafter, they correctly sketched the corresponding graph as shown in Figure 20.

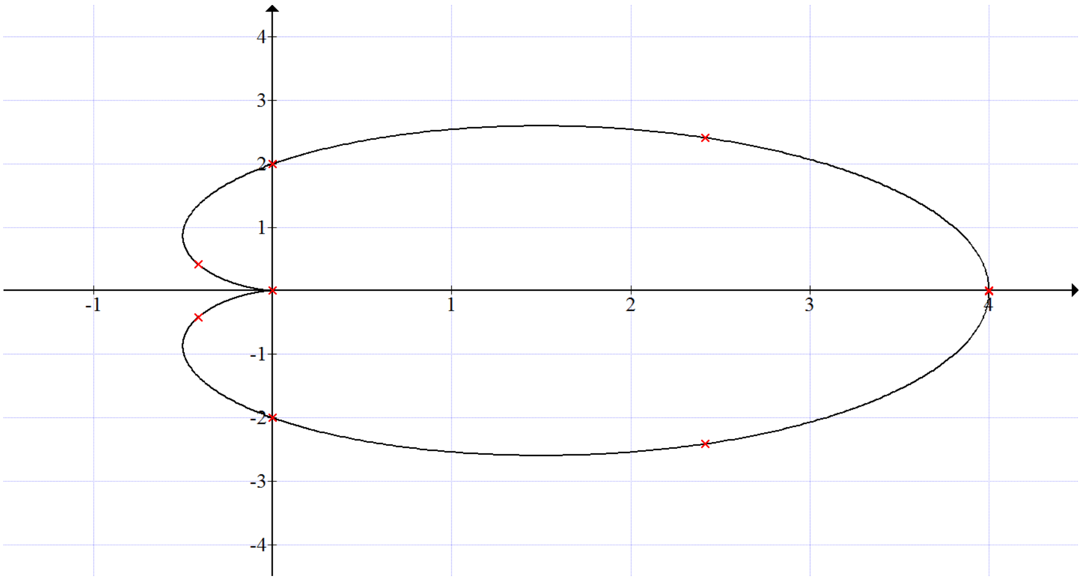
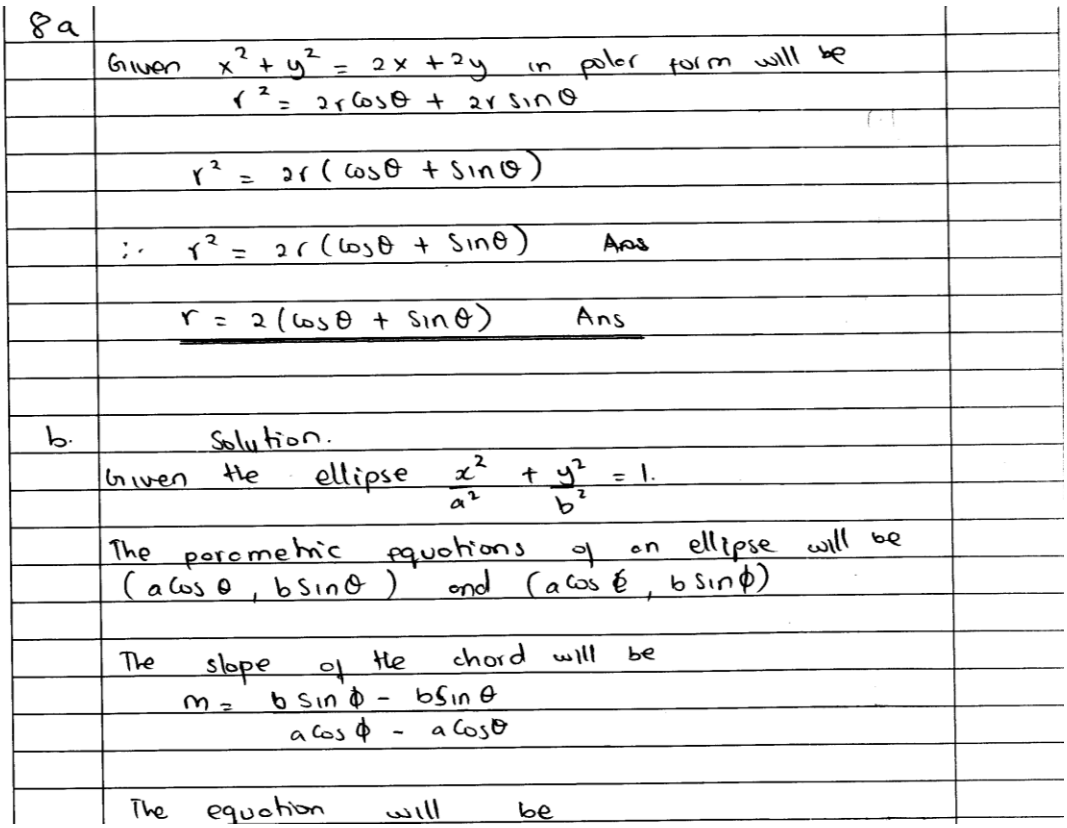


Figure 20: *The graph of* $r = 2(1+\cos\theta)$ *against* q

Extract 18.1 is a sample of candidates' correct responses to question 8.



8a

Given $x^2 + y^2 = 2x + 2y$ in polar form will be

$r^2 = 2r\cos\theta + 2r\sin\theta$

$r^2 = 2r(\cos\theta + \sin\theta)$

$\therefore\ r^2 = 2r(\cos\theta + \sin\theta)$ Ans

$r = 2(\cos\theta + \sin\theta)$ Ans

b. Solution.

Given the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

The parametric equations of an ellipse will be $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$

The slope of the chord will be

$m = \frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta}$

The equation will be

8b

$$\frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$\frac{b(\sin\phi - \sin\theta)}{a(\cos\phi - \cos\theta)} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$\frac{b\,(2\cos\frac{1}{2}(\phi+\theta)\sin\frac{1}{2}(\phi-\theta))}{a\,(-2\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\phi-\theta))} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$-\frac{b}{a}\frac{\cos\frac{1}{2}(\theta+\phi)}{\sin\frac{1}{2}(\theta+\phi)} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$-xb\cos\tfrac{1}{2}(\theta+\phi) + ab\cos\theta\cos\tfrac{1}{2}(\theta+\phi) = ya\sin\tfrac{1}{2}(\theta+\phi) - ab\sin\tfrac{1}{2}(\theta+\phi)\sin\theta$$

$$ab\,(\cos\theta\cos\tfrac{1}{2}(\theta+\phi) + \sin\tfrac{1}{2}(\theta+\phi)\sin\theta) = xb\cos\tfrac{1}{2}(\theta+\phi) + ya\sin\tfrac{1}{2}(\theta+\phi)$$

$$ab\,(\cos(\theta - \tfrac{1}{2}\theta - \tfrac{\phi}{2})) = xb\cos\tfrac{1}{2}(\theta+\phi) + ya\sin\tfrac{1}{2}(\theta+\phi)$$

$$ab\cos(\tfrac{1}{2}\theta - \tfrac{\phi}{2}) = xb\cos\tfrac{1}{2}(\theta+\phi) + ay\sin\tfrac{1}{2}(\theta+\phi)$$

The equation of the chord will be

$\therefore$

$$xb\cos\tfrac{1}{2}(\theta+\phi) + ay\sin\tfrac{1}{2}(\theta+\phi) = ab\cos\tfrac{1}{2}(\theta-\phi) \quad \text{Ans}$$

8c

To show that $(a\sec\theta, b\tan\theta)$ lies on the hyperbola is the same as showing

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Using the L.H.S of the equation above

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(a\sec\theta)^2}{a^2} - \frac{(b\tan\theta)^2}{b^2}$$

$$= \frac{a^2\sec^2\theta}{a^2} - \frac{b^2\tan^2\theta}{b^2}$$

$$= \sec^2\theta - \tan^2\theta$$

from the trigonometry identity $\sec^2\theta - \tan^2\theta = 1$

Hence $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Hence shown) Ans

To obtain the equation of the tangent at P from P (a sec θ , b tan θ)

$x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b\sec^2\theta$

$\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right)$

$= b\sec^2\theta \cdot \frac{1}{a\sec\theta\tan\theta}$

$\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$

$= \frac{b}{a}\frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}$

$\frac{dy}{dx} = \frac{b}{a\sin\theta}$

but equation of tangent

$\frac{b}{a\sin\theta} = \frac{y - b\tan\theta}{x - a\sec\theta}$

$bx - ab\sec\theta = ay\sin\theta - ab\tan\theta\sin\theta$

$bx - ab\left(\frac{1}{\cos\theta}\right) = ay\sin\theta - ab\left(\frac{\sin\theta}{\cos\theta}\right)\sin\theta$

$bx\cos\theta - ab = ay\sin\theta\cos\theta - ab\sin^2\theta$

$bx\cos\theta - ay\sin\theta\cos\theta = ab - ab\sin^2\theta$

$bx\cos\theta - ay\sin\theta\cos\theta = ab\cos^2\theta$

$bx - ay\sin\theta = ab\cos\theta$

∴ The equation of tangent is

$bx - ay\sin\theta = ab\cos\theta$ Ans

d Solution.

from $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

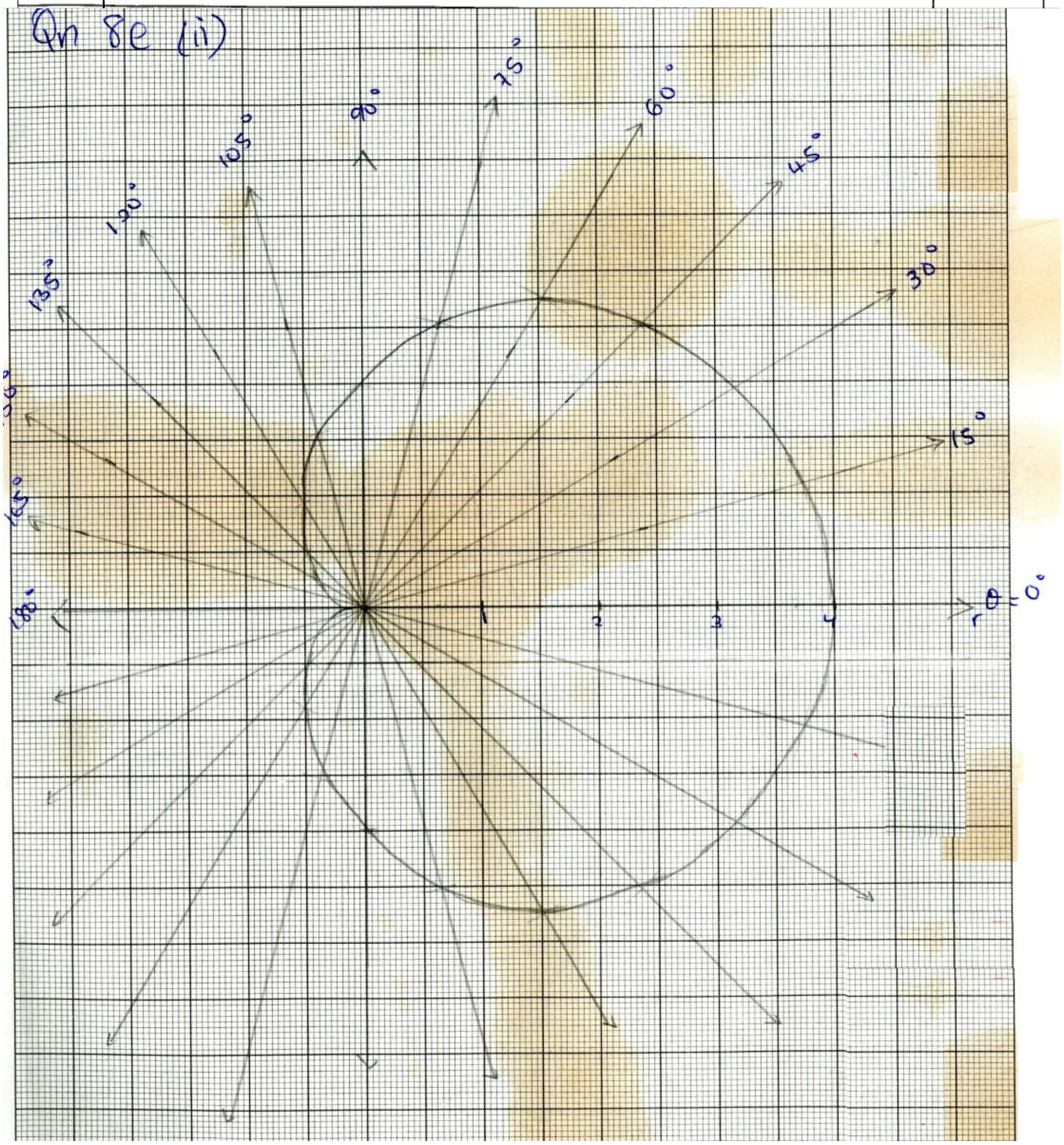
	The slope of normal.	
	$m_1 m_2 = -1$	
8d.	$m_2 = \frac{-1}{m_1}$	
	$= -1\left(\frac{y}{2a}\right)$	
	$= \frac{-y}{2a}$	
	$m_{normal} = \frac{-y_1}{2a}$	
	The equation will be	
	$\frac{-y_1}{2a} = \frac{y - y_1}{x - x_1}$	
	$xy_1 - x_1y_1 = -2ay + 2ay_1$	
	$xy_1 - x_1y_1 + 2ay - 2ay_1 = 0$	
	$(x - x_1)y_1 + 2a(y - y_1) = 0$ Hence shown	
	$\therefore (x - x_1)y_1 + 2a(y - y_1) = 0$ Hence shown	
e.	from	
	$r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$	
	$b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$	
	$b^2(r\cos\theta)^2 + a^2(r\sin\theta)^2 = a^2b^2$	
	but $x = r\cos\theta$ and $y = r\sin\theta$	
	$b^2x^2 + a^2y^2 = a^2b^2$	
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$	
	$\therefore$ The cartesian equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Ans	
8e ii)	Required to draw	
	$r = 2(1 + \cos\theta)$	
	The given curve is symmetrical about the initial line $\theta = 0°$	
	At tangency point	
	$r = 0.$	
	$0 = 2(1 + \cos\theta)$	
	$0 = 1 + \cos\theta$	

$\cos\theta = -1$

$\theta = \cos^{-1}(-1)$

$\theta = 180^\circ$

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
r	4	3.9	3.7	3.4	3	2.5	2	1.5	1	0.6	0.3	0.06	0



Extract 18.1: *A sample of correct response to Question 8*

In Extract 18.1, the candidate was able to change the Cartesian equation into polar form and vice versa, find the equation of the chord from the given eccentric angles and use the concepts of tangents and normal to the curve to find the corresponding equations. The candidate was also able to sketch the graph of the given polar curve.

Despite candidates performing well in this question, there were candidates (11%) who scored low marks due to the following reasons: In part (a), some candidates transformed the equation $x^2 + y^2 = 2x + 2y$ into $\left(\frac{x-1}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$ contrary to the requirement of the question.

In part (b), a number of candidates managed to formulate the parametric coordinates in terms of eccentric angles θ and ϕ i.e. $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$ but failed to find the equation of the chord on the ellipse. Several candidates for instance, differentiated the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to get $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$. Thereafter they substituted the coordinates $x = a\cos\theta$ and $y = b\sin\theta$ into $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$ to get the slope of the chord as $\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$. Then, they carried out the procedures of finding the equation of a line to get $2ay\sin\theta = 2b\cos\theta(x-a)(\cos\theta+\cos\phi) + \mathrm{a}(\sin\theta+\sin\phi)$. A number of these candidates could not create the coordinates of the chord $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$ by using the given eccentric angles. These candidates considered these angles as the coordinates (θ, ϕ) of the chord on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Due to this reason, they committed the following mistakes: firstly, they differentiated the standard equation of the ellipse to get the slope $\frac{dy}{dx} = -\frac{\theta b^2}{\phi a^2}$; secondly, they substituted the slope obtained in step one into the equation $y = m(x - x_0) + y_0$ to get incorrect equations of the chord such as $qa^2x + fb^2y = q^2a^2 + f$. Other candidates in this category did not present the equation of the chord in the form $y = mx + c$ or $ax + by + c = 0$. One candidate for instance, quoted the equation of the chord as $\cos^2(\theta+\phi) + \sin^2(\theta+\phi) = 1$ which was wrong.

In part (c), the analysis of the candidates' responses shows that most candidates had insufficient knowledge of differentiation. For example, several of these candidates wrote the derivative of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as $2x - 2y\frac{dy}{dx} = 1$ instead of $2x - 2y\frac{dy}{dx} = 0$. Due to this reason, they got incorrect equations of tangent to the hyperbola like $-2y^3a + 2aby^2 \tan\theta = (\mathrm{x} - \mathrm{a}\sec\theta)(\mathrm{b}^2\mathrm{a}^2 - 2\mathrm{b}^2\mathrm{x})$. Furthermore, a number of these candidates substituted the gradient of the tangent $\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$ into the formula $m_1m_2 = -1$ to get $\frac{dy}{dx} = -\frac{a\tan\theta}{b\sec\theta}$ which produced the equation of the tangent to be $ax\tan\theta + by\sec\theta = \left(a^2 + b^2\right)\sec\theta\tan\theta$ instead of $bx\sec\theta - ay\tan\theta = ab$.

In part (d), most candidates tried to find the equation of the normal to the parabola by using the slope of the tangent which is $\frac{dy}{dx} = \frac{2a}{y_1}$. Hence, they failed to get $\left(x - x_1\right)y_1 + 2a\left(y - y_1\right) = 0$ as per instruction of this question.

The candidates who failed to answer part (e) (i) correctly substituted $x = r\sin q$ and $y = r\cos q$ to produce $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ instead of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In part (e) (ii), the weake candidates had difficulties in creating the table of values which was a necessary step for sketching the polar graph, as a result they ended up sketching incorrect graphs. The weak candidates also had difficulties in using the table of values to sketch the required graph. Extract 18.2 shows a sample solution from a candidate who answered part (a) of this question incorrectly.

8. e).

$$r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2$$

$$r^2 = x^2+y^2$$

$$x = a\cos\theta$$

$$y = a\sin\theta$$

$$(x^2+y^2)\left(b^2\left(\frac{x^2}{a^2}\right) + a^2\left(\frac{y^2}{a^2}\right)\right) = a^2b^2$$

$$(x^2+y^2)\left(\frac{bx^2}{a^2} + \frac{y^2}{1}\right) = a^2b^2$$

$$(x^2+y^2)\left(\frac{bx^2+ay^2}{a^2}\right) = a^2b^2$$

$$(x^2+y^2)(bx^2+ay^2) = a^4b^2$$

$$bx^4 + ax^2y^2 + by^2x^2 + ay^4 = a^4b^2$$

$$bx^4 + ax^2y^2 + by^2x^2 + ay^4 = a^4b^2$$

$$bx^4 + (a+b)x^2y^2 + ay^4 = a^4b^2$$

Extract 18.2: *A sample of an incorrect response to Question 8 (a)*

In Extract 18.2, the candidate failed to define the Cartesian coordinates in polar form in part (a).

3.0 ANALYSIS OF CANDIDATES' PERFORMANCE IN EACH TOPIC

The examination tested 18 topics out of which 10 topics were in paper one and eight topics in paper two. The topics which were tested in paper one are *Calculating Devices*, *Hyperbolic Functions*, *Linear Programming*, *Statistics*, *Sets*, *Functions*, *Numerical Methods*, *Coordinate Geometry I*, *Integration* and *Differentiation*. The topics which were tested in paper two are *Probability*, *Logic*, *Vectors*, *Complex Numbers*, *Trigonometry*, *Algebra*, *Differential Equations* and *Coordinate Geometry II*.

The analysis of data in each topic shows that out of the 18 topics that were tested 15 topics had good performance. These topics were *Sets*, *Numerical Methods*, *Linear Programming*, *Coordinate Geometry II*, *Functions*, *Hyperbolic Functions*, *Complex Numbers*, *Trigonometry*, *Statistics*, *Logic*, *Calculating Devices*, *Algebra*, *Differential Equations*, *Coordinate Geometry I* and *Probability*. The 2020 statistical data analysis is similar to that of 2021 where 15 topics had good performance.

Good performance in these topics was due to candidates' ability to do the following:

(a) Simplify set expressions and present information using Venn diagrams;

(b) Apply Simpson's and Trapezoidal rule in computing the area of curves;

(c) Solve linear programming problems graphically;

(d) Find equations of tangent and normal to various conic sections;

(e) Draw graph of polar equations, composite and rational functions;

(f) Convert inverse hyperbolic cosine into logarithmic functions;

(g) differentiate hyperbolic functions.

(h) Test the validity of an argument by using laws of algebra of propositions and truth tables;

(i) Represent the compound statements with a network diagram and vice versa;

(j) Apply Cramer’s rule in solving 3 by 3 system of linear equations;

(k) use scientific calculators to perform computation of various mathematical expressions.

(l) apply differential equations in solving a real life problem; and

(m) A high degree of carefulness and avoiding careless mistakes, blunders and errors.

Despite the candidates’ good performance, the topic of Vectors (43.3%) had average performance.. The topics were Integration (33.9%) and Differentiation (25.2%) had weak performance. In 2020, the topics of Probability (44.3%) and Vectors (57.9 %) had average performance while the topic of Coordinate Geometry I (18.2%) was poorly performed. See appendix I and II.

The poor performance in Integration and Differentiation was due to failure of candidates to do the following:

(a) Use integration by parts method in integrating the product of algebraic and trigonometric functions;

(b) Use the substitution rule to evaluate definite integrals;

(c) Use turning points and intercepts in sketching the graph of polynomial functions; and to

(d) Find the Taylor’s series of radical functions.

The comparison of the candidates’ performance in each topic for three consecutive years is shown in appendix III.

4.0 CONCLUSION AND RECOMMENDATIONS

4.1 Conclusion

This report has shown that 93.49 per cent of candidates passed the examination in 2021, compared to 90.63 per cent who passed the examination in 2020. The report shows that 15 topics were performed well, 01 topic had average performance and 02 topics were performed poorly. The candidate's poor performance was due to: failure of candidates to: use the integration by parts method in integrating the product of algebraic and trigonometric functions; use correctly the substitution rule to compute definite integrals; use turning points in sketching the graph of polynomial functions and find the Taylor's series of radical functions.

It is expected that the stakeholders will make use of the recommendations in this report to enhance performance in future Advanced Mathematics Examinations.

4.2 Recommendations

For the purpose of improving the candidates' performance in this subject especially in the topics of Integration and Differentiation, it is suggested that:

(a) Students should be encouraged to study in groups so as to improve their capacity in using techniques of integration by parts to integrate the product of two expressions involving algebra and trigonometric functions;

(b) Teachers should guide students in small groups on how to apply the technique of substitution in calculating the value of the definite integral;

(c) Teachers should use questions and answers to lead students to draw graphs of various functions especially the graph of polynomial functions by using intercepts, turning points as well as the teaching and learning resources such as graph boards, graph papers and flip charts;

(d) Teachers should guide students how to use Taylor's theorem to expand various functions, especially those involving fractional exponents by using teaching and learning tools such as the chart of Taylor's series; and

(e) The Ministry of Regional Administration and Local Government should conduct in-service training to teachers on topics related to differentiation and integration.

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Appendix I

Analysis of Candidates' Performance in each Topic in the 2021 Advanced Mathematics Examination

S/N	Topic	Question Number	The Percentage of Candidates who Passed	Remarks
1	Sets	5	91.2	Good
2	Numerical Methods	7	89.8	Good
3	Linear Programming	3	89.7	Good
4	Coordinate Geometry II	8	88.8	Good
5	Functions	9	88.1	Good
6	Hyperbolic Functions	2	86.8	Good
7	Complex Numbers	4	82.4	Good
8	Trigonometry	5	82.2	Good
9	Statistics	4	80.3	Good
10	Logic	2	79.4	Good
11	Calculating Devices	1	70.7	Good
12	Algebra	6	68.1	Good
13	Differential Equations	7	68.1	Good
14	Coordinate Geometry I	8	65.7	Good
15	Probability	1	62.1	Good
16	Vectors	3	43.3	Average
17	Integration	9	33.9	Weak
18	Differentiation	10	25.2	Weak

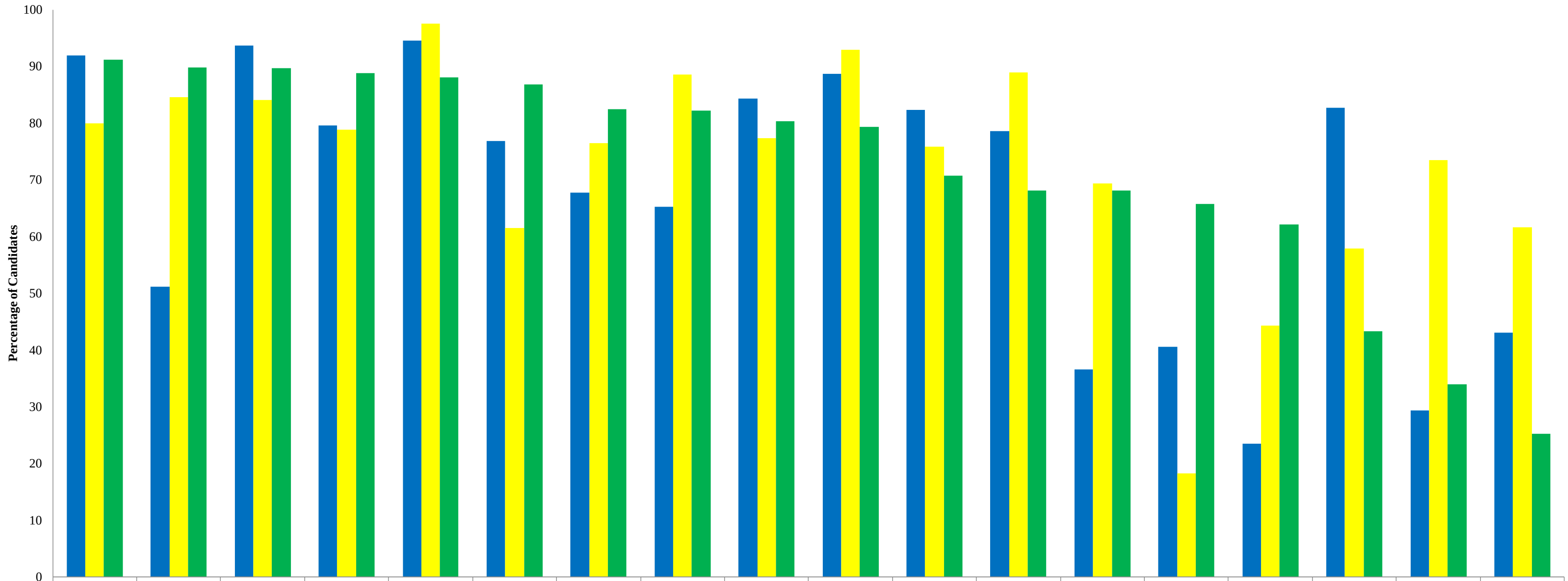
Appendix II

Analysis of Candidates' Performance in each Topic in the 2020 and 2021 Advanced Mathematics Examination

S/N	Topic	2020			2021	
		Question Number	The Percentage of Candidates who Passed	Remarks	The Percentage of Candidates who Passed	Remarks
1	Sets	5	80.0	Good	91.2	Good
2	Numerical Methods	7	84.6	Good	89.8	Good
3	Linear Programming	3	84.1	Good	89.7	Good
4	Coordinate Geometry II	8	78.8	Good	88.8	Good
5	Functions	9	97.5	Good	88.1	Good
6	Hyperbolic Functions	2	61.5	Good	86.8	Good
7	Complex Numbers	4	76.5	Good	82.4	Good
8	Trigonometry	5	88.6	Good	82.2	Good
9	Statistics	4	77.3	Good	80.3	Good
10	Logic	2	92.9	Good	79.4	Good
11	Calculating Devices	1	75.8	Good	70.7	Good
12	Algebra	6	89	Good	68.1	Good
13	Differential Equations	7	69.4	Good	68.1	Good
14	Coordinate Geometry I	8	18.2	Weak	65.7	Good
15	Probability	1	44.3	Average	62.1	Good
16	Vectors	3	57.9	Average	43.3	Average
17	Integration	9	73.5	Good	33.9	Weak
18	Differentiation	10	61.6	Good	25.2	Weak

Appendix III

The candidates' performance in Advanced Mathematics topic-wise for three consecutive years



	Sets	Numerical Methods	Linear Programming	Coordinate Geometry II	Functions	Hyperbolic Functions	Complex Numbers	Trigonometry	Statistics	Logic	Calculating Devices	Algebra	Differential Equations	Coordinate Geometry I	Probability	Vectors	Integration	Differentiation
2019	91.9	51.1	93.7	79.6	94.5	76.9	67.8	65.3	84.3	88.7	82.3	78.6	36.6	40.6	23.4	82.7	29.3	43
2020	80	84.6	84.1	78.8	97.5	61.5	76.5	88.6	77.3	92.9	75.8	89	69.4	18.2	44.3	57.9	73.5	61.6
2021	91.2	89.8	89.7	88.8	88.1	86.8	82.4	82.2	80.3	79.4	70.7	68.1	68.1	65.7	62.1	43.3	33.9	25.2